6 SINGLE-ECHELON SYSTEMS: INTEGRATION - OPTIMALITY

6.1 Joint optimization of order quantity and reorder point

In practice it is most common to determine the batch quantity from a deterministic model. The stochastic demand is then replaced by its mean. In Chapter 4 we have considered different methods for determination of batch quantities under the assumption of deterministic demand. Stochastic variations in the demand, and possibly in the lead-time, are then only taken into account when determining the reorder point. As discussed in Chapter 4 this procedure is, in general, an adequate approximation. In Chapter 5 we have described various techniques for determining the reorder point for a given batch quantity.

It is also possible, though, to optimize the batch quantity and the reorder point jointly in a stochastic model. In this section we shall consider such techniques.

6.1.1 Discrete demand

Assume discrete compound Poisson demand. Each customer demands an integral number of units. As before, we also assume that not all demands are multiples of some integer larger than one. The average demand per unit of time is denoted μ. The lead-time L is constant. The stochastic lead-time demand is denoted $D(L)$ and its mean $\mu' = \mu L$.

Furthermore, we consider a holding cost h per unit and time unit, a shortage cost b_1 per unit and time unit, and an ordering or setup cost A.

6.1.1.1 (*R*, *Q*) policy

We shall first deal with a continuous review (R, Q) policy and the joint optimization of the integers R and Q. A periodic review model can be handled in essentially the same way. See also Federgruen and Zheng (1992).

Recall the following standard argument from Section 5.3.2. Let $IP(t)$ be the inventory position at time t. Consider then the time $t + L$. At that time, everything that was on order at time t has been delivered. Orders that have been triggered in the interval $(t, t + L]$ have not reached the inventory due to the lead-time. Consequently we have

$$IL(t + L) = IP(t) - D(L). \tag{6.1}$$

Let us initially consider the special case of an $(S - 1, S)$ policy with $S = k$, i.e., $R = k - 1$ and $Q = 1$. This means that the inventory position is k at all times. Using (6.1) the inventory level distribution can then be obtained as

$$P(IL = j) = P(D(L) = k - j), \qquad j \le k. \tag{6.2}$$

Let $g(k)$ be the average holding and shortage costs per time unit. We have from (5.56)

$$g(k) = -b_1 E(IL) + (h + b_1)E(IL^+) = -b_1(k - \mu') + (h + b_1)\sum_{j=1}^{k} jP(IL = j). \tag{6.3}$$

The results in Section 5.9.1 imply that $g(k)$ is a convex function of the inventory position k. Furthermore $g(k) \to \infty$ as $|k| \to \infty$.

Let us now go back to the (R, Q) policy. Recall that the inventory position is uniform on $[R + 1, R + Q]$. Consequently the total average costs per time unit can be expressed as

$$C(R, Q) = \frac{A\mu}{Q} + \frac{1}{Q}\sum_{k=R+1}^{R+Q} g(k). \tag{6.4}$$

In (6.4) we obtain the holding and backorder costs by averaging over the inventory position. We assume that each batch incurs an ordering cost A, i.e.,

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نسبت به قیمت پیشنهادی پرداخت میشود.

if an order for two batches is triggered, the associated ordering cost is $2A$. Our objective is to optimize $C(R, Q)$ with respect to both R and Q.

Let us now define $C(Q)$ as

$$C(Q) = \min_R \{C(R, Q)\}. \tag{6.5}$$

It is evident that $C(1) = A\mu + \min_k \{g(k)\}$, i.e., if the sum in (6.4) includes a single value of k, we choose a k that gives the minimum cost. We denote an optimal k, i.e., a value of k that minimizes $g(k)$ by k^*. A corresponding optimal reorder point for $Q = 1$ is $R^*(1) = k^* - 1$, i.e., if we use a single value of k, we choose the best one. Consider now $Q = 2$, which means that we use two values of k. Due to the convexity of $g(k)$, the second best k must be either $k^*- 1$ or $k^*+ 1$. Clearly, we should use the better of these two values. From (6.4) it is therefore evident that it is optimal to choose $R^*(2) = R^*(1) - 1$ if $g(R^*(1)) \leq g(R^*(1) + 2)$, and $R^*(2) = R^*(1)$ otherwise. We obtain $C(2) = A\mu/2 + [\min\{g(R^*(1)), g(R^*(1) + 2)\} + g(R^*(1) + 1)]/2$, or equivalently, $C(2) = C(1)/2 + \min\{ g(R^*(1)), g(R^*(1) + 2)\}/2$. More generally we have

$$\begin{aligned} &R^*(Q+1) = R^*(Q) - 1 \quad \textit{if } g(R^*(Q)) \leq g(R^*(Q) + Q + 1), \\ &R^*(Q+1) = R^*(Q) \quad \text{otherwise,} \end{aligned} \tag{6.6}$$

and

$$C(Q+1) = C(Q)\frac{Q}{Q+1} + \left[\min\left\{g(R^*(Q), g(R^*(Q) + Q + 1)\right\}\right]\frac{1}{Q+1}. \tag{6.7}$$

It is evident from (6.7) that $C(Q + 1) \geq C(Q)$ if and only if $\min\{g(R^*(Q), g(R^*(Q) + Q + 1)\} \geq C(Q)$. Furthermore, it is obvious that $\min\{g(R^*(Q), g(R^*(Q) + Q + 1)\}$ is increasing with Q. Let Q^* be the smallest Q such that $C(Q + 1) \geq C(Q)$. It follows from (6.7) that $C(Q) \geq C(Q^*)$ for any $Q \geq Q^*$. Consequently, Q^* and $R^*(Q^*)$ provide the optimal solution.

To summarize, it is very easy to determine the optimal solution by applying (6.6) and (6.7) until the costs increase.

6.1.1.2 (*s*, *S*) policy

Let us now instead consider the optimization of an (*s*, *S*) policy. Otherwise the assumptions are exactly the same as above. When using an (*s*, *S*) policy the inventory position is no longer uniformly distributed. This makes the optimization more complex. Zheng and Federgruen (1991), however, have developed a very efficient optimization procedure. We shall here only describe the procedure and refer to their paper for proofs and more details.

In Section 5.11 we defined and determined the probabilities

$m_j =$ probability to reach $IP = j$ during an order cycle ($s + 1 \leq j \leq S$).

The average total number of customers during an order cycle is $\sum_{j=s+1}^{S} m_j$. Let λ be the customer arrival rate. The average length of an order cycle is consequently $\sum_{j=s+1}^{S} m_j / \lambda$. The steady state distribution of the inventory position is obtained as

$$P(IP = k) = m_k \Big/ \sum_{j=s+1}^{S} m_j, \quad k = s+1, s+2, ..., S. \tag{6.8}$$

See Section 5.11 for more details. Given these probabilities we can determine the average costs per time unit as

$$C(s,S) = \frac{A\lambda}{\sum_{j=s+1}^{S} m_j} + \sum_{k=s+1}^{S} P(IP = k) \cdot g(k). \tag{6.9}$$

We are now ready to describe the optimization procedure.

1. In the first step we set $S^* = k^*$, i.e., a value of k that minimizes $g(k)$. Next, consider $s = S^* - 1$, $s = S^* - 2$, ... , until $C(s, S^*) \leq g(s)$. When this occurs, set $s^* = s$ and the initial best solution as $C^* = C(s^*, S^*)$. Set $S = S^*$.

2. Set $S = S + 1$. If $g(S) > C^*$, s^* and S^* provide the optimal solution with the costs C^*, and the algorithm stops.

3. It is possible to show that the considered S will improve the solution if, and only if, $C(s^*, S) < C^*$. In that case set $S^* = S$. Otherwise go to 2.

4. To find the best s corresponding to S^*, it is only necessary to consider $s = s^*, s^* + 1, \ldots$ The new s^* is obtained as the smallest value of s giving $C(s, S^*) > g(s + 1)$. Update $C^* = C(s^*, S^*)$ and go to 2.

6.1.2 An iterative technique

Now consider normally distributed demand instead, and add ordering costs to the (R, Q) model with backorder cost per unit and unit time that we dealt with in Section 5.9.2. The lead-time demand has mean μ' and standard deviation σ'. The mean per unit of time is μ.

By adding the average ordering costs per time unit to the cost expression in (5.65) we have

$$C(R,Q) = h(R + Q/2 - \mu') + (h + b_1)\frac{\sigma'^2}{Q}\left[H\left(\frac{R-\mu'}{\sigma'}\right) - H\left(\frac{R+Q-\mu'}{\sigma'}\right)\right] + \frac{A\mu}{Q}. \tag{6.10}$$

Recall the definitions of $H(x)$ and $G(x)$. See Appendix 2.

Our objective is to optimize $C(R, Q)$ with respect to R and Q. As pointed out in Section 5.9.2, we cannot replace the backorder costs by the fill rate (5.67) when carrying out a joint optimization of R and Q.

We shall demonstrate how the optimization can be carried out by a simple iterative procedure. It can be shown that this procedure will always converge to the optimal solution (Rosling, 2002b).

The necessary conditions $\partial C / \partial Q = \partial C / \partial R = 0$ are also sufficient and will guarantee the unique optimal solution.

We obtain $\partial C / \partial R$ as in (5.66)

$$\frac{\partial C}{\partial R} = h + (h + b_1)\frac{\sigma'}{Q}\left[G\left(\frac{R+Q-\mu'}{\sigma'}\right) - G\left(\frac{R-\mu'}{\sigma'}\right)\right]. \tag{6.11}$$

For a given Q we can use (6.11) to determine the corresponding reorder point R giving $\partial C / \partial R = 0$. The resulting R decreases with Q. We can also get $\partial C / \partial Q$ from (6.10) as

$$\frac{\partial C}{\partial Q} = \frac{h}{2} - \frac{A\mu}{Q^2}$$

$$-(h+b_1)\frac{\sigma'^2}{Q^2}\left[H\left(\frac{R-\mu'}{\sigma'}\right) - H\left(\frac{R+Q-\mu'}{\sigma'}\right) - \frac{Q}{\sigma'}G\left(\frac{R+Q-\mu'}{\sigma'}\right)\right].$$

(6.12)

We shall now describe the iterative procedure for finding the optimal solution satisfying $\partial C / \partial Q = \partial C / \partial R = 0$.

We start by determining the batch quantity according to the classical economic order quantity model

$$Q^0 = \sqrt{2A\mu / h}\ . \tag{6.13}$$

Next we determine the corresponding reorder point R^0 from (6.11) and the condition $\partial C / \partial R = 0$. In the following step we get a new batch quantity Q^1 from

$$Q^{i+1} = \left[\frac{2A\mu}{h} + \frac{2(h+b_1)}{h}\sigma'^2\left(H\left(\frac{R^i-\mu'}{\sigma'}\right) - H\left(\frac{R^i+Q^i-\mu'}{\sigma'}\right)\right.\right.$$

$$\left.\left. - \frac{Q^i}{\sigma'}G\left(\frac{R^i+Q^i-\mu'}{\sigma'}\right)\right)\right]^{1/2}.$$

(6.14)

After that we determine the reorder point R^1 corresponding to Q^1 from (6.11) and the condition $\partial C / \partial R = 0$. Given Q^1 and R^1, we obtain Q^2 and R^2 in the same way, etc.

It can be shown that the batch quantity increases in each step, $Q^{i+1} \geq Q^i$, while the reorder point decreases, $R^{i+1} \leq R^i$. The costs decrease in each step. Let C^* be the optimal cost. It is possible to show that $C(R^i, Q^{i+1})$-$C^* \leq h(Q^{i+1} - Q^i)$, i.e., the remaining gap can be bounded.

Example 6.1 Let $A = 100$, $h = 2$, and $b_1 = 20$. The demand per time unit is normally distributed with $\mu = 50$ and $\sigma = 20$. The lead-time is $L = 4$. We obtain $\mu' = \mu L = 200$ and $\sigma' = \sigma L^{1/2} = 40$.

The results from the iterations when applying the described procedure are shown in Table 6.1. We can see that the costs converge very rapidly. The changes in batch quantity and reorder point as compared to the initial solution are significant. Still the total cost reduction is only about 2.5 percent.

Table 6.1 Results from the iterations for the data in Example 6.1.

Iteration i	0	1	2	3	4	5
Order quantity Q^i	70.71	87.91	93.08	94.59	95.03	95.15
Reorder point R^i	224.76	219.60	218.16	217.75	217.63	217.60
Costs C^i	232.01	226.63	226.24	226.21	226.20	226.20

Similar procedures for models with other types of costs can be designed in the same way. We could, for example, have used a shortage cost per unit as in Section 5.10. A corresponding procedure for a model with lost sales is given in Rosling (2002a).

6.1.3 Fill rate constraint - a simple approach

We shall now consider a different and very simple technique from Axsäter (2004). This technique is especially suitable when optimizing R and Q under a fill rate constraint, because it is in general only necessary to consider relatively few different fill rates. The cost function is the same as in (6.10) except for the backorder costs that are omitted

$$C = h(R + Q/2 - \mu') + h\frac{\sigma'^2}{Q}\left[H\left(\frac{R-\mu'}{\sigma'}\right) - H\left(\frac{R+Q-\mu'}{\sigma'}\right)\right] + \frac{A\mu}{Q}. \tag{6.15}$$

The fill rate is according to (5.52) obtained as

$$S_2 = 1 - \frac{\sigma'}{Q}\left[G\left(\frac{R-\mu'}{\sigma'}\right) - G\left(\frac{R+Q-\mu'}{\sigma'}\right)\right]. \tag{6.16}$$

It is possible to show that the considered problem has a pure optimal strategy, i.e., a single (R, Q) as its optimal solution, see Rosling (2002b).

At a first glance the problem to optimize (6.15) under the constraint (6.16) for a given fill rate S_2 depends on five parameters: h, A, μ, μ', and σ'. However, it is easy to see that the problem does in fact depend on a single parameter only. Define

$$\begin{aligned} c &= C/(h\sigma'), \\ q &= Q/\sigma', \\ r &= (R - \mu')/\sigma', \\ E &= A\mu/(h(\sigma')^2). \end{aligned}$$

Substituting in (6.15) and (6.16) we get the equivalent problem to minimize

$$c = r + \frac{q}{2} + \frac{1}{q}\left[H(r) - H(r+q)\right] + \frac{E}{q}, \qquad (6.17)$$

under the constraint

$$S_2 = 1 - \frac{1}{q}\left[G(r) - G(r+q)\right]. \qquad (6.18)$$

Note that this version of the problem for a certain given S_2 depends on a single problem parameter, $E \geq 0$.

Axsäter (2004) suggests that q^* is determined by linear interpolation of tabulated values (or by using a polynomial approximation). (The tabulated values can, for example, be obtained by using the technique in Section 6.1.2.) The corresponding r^* is then obtained from (6.18) in a second step. A part of the required table is given in Table 6.2.

We are now ready to describe a simple way to solve the original problem for a given fill rate S_2 and any problem parameters. We start by determining E. Next we obtain the solution of the one-parameter problem from the table and (6.18). Given the optimal solution q^* and r^*, we get the solution of the original problem as

$$\begin{aligned} Q^* &= q^*\sigma', \\ R^* &= r^*\sigma' + \mu'. \end{aligned}$$

Note that we can use the same table repeatedly for all items.

Example 6.2 Let $S_2 = 0.9$, $A = 100$, $h = 2$, $\mu = 50$, $\mu' = 200$, and $\sigma' = 40$, i.e., except for the backorder cost, the same data as in Example 6.1. We get $E = A\mu/(h(\sigma')^2) = 1.5625$ and $e = \ln(E) = 0.4463$. Using the table values for $e = 0.4$ and $e = 0.5$ and interpolating linearly, we get $q^* = (2.5111(0.5 - 0.4463) + 2.6070(0.4463 - 0.4))/0.1 = 2.5555$. Using (6.18) we get $r^* = 0.3294$. Finally we obtain $Q^* = q^*\sigma' = 2.5555 \cdot 40 = 102.22$ and $R^* = r^*\sigma' + \mu' = 0.3294 \cdot 40 + 200 = 213.18$. The corresponding optimal solution is $Q^* = 102.20$ and $R^* = 213.14$.

Table 6.2 q^* for different fill rates and values of $e = ln(E)$.

e \ S	60%	70%	80%	85%	90%	95%	99%
-0.2	2.7398	2.4609	2.2323	2.1255	2.0165	1.8926	1.7371
-0.1	2.8408	2.5496	2.3127	2.2025	2.0904	1.9633	1.8047
0.0	2.9462	2.6421	2.3964	2.2828	2.1675	2.0373	1.8756
0.1	3.0562	2.7383	2.4836	2.3664	2.2478	2.1145	1.9498
0.2	3.1712	2.8387	2.5745	2.4536	2.3318	2.1953	2.0276
0.3	3.2914	2.9435	2.6694	2.5446	2.4194	2.2798	2.1092
0.4	3.4172	3.0529	2.7684	2.6397	2.5111	2.3683	2.1949
0.5	3.5490	3.1671	2.8718	2.7391	2.6070	2.4609	2.2848
0.6	3.6872	3.2867	2.9800	2.8430	2.7073	2.5580	2.3791
0.7	3.8322	3.4118	3.0931	2.9518	2.8124	2.6599	2.4783
0.8	3.9846	3.5430	3.2116	3.0658	2.9226	2.7668	2.5824

6.2 Optimality of ordering policies

In all the models that we have considered in this chapter it has been assumed that the policy is either of the (R, Q) type or of the (s, S) type. A natural question to ask is whether other, better policies exist. This is, in general, not the case. In most situations one of these policies is indeed optimal for a single-echelon inventory system with independent items.

In Section 6.2.1 we shall show that an (R, Q) policy is optimal when there are no ordering costs but a given fixed batch quantity Q. After that we comment on the optimality of (s, S) policies in Section 6.2.2.

6.2.1. Optimality of (R, Q) policies when ordering in batches

Consider an inventory system with continuous review. Assume discrete compound Poisson demand and that each customer demands an integral number of units. As before, we assume that not all demands are multiples of some integer larger than one. The average demand per unit of time is denoted μ. The lead-time L is constant. The stochastic lead-time demand is denoted $D(L)$ and its mean $\mu' = \mu L$.

Furthermore, we consider a holding cost h per unit and time unit and a shortage cost b_1 per unit and time unit. There are no ordering costs but all orders must be multiples of a given batch quantity Q. Orders can only be triggered by customer demands. We shall show that an (R, Q) policy is optimal under these assumptions. Our proof follows essentially Chen (2000).

Obviously the inventory position must be an integer at all times. Assume first that the inventory position is k at some arbitrary time t. Using (6.1) the inventory level distribution at time $t + L$ can be obtained as

$$P(IL = j) = P(D(L) = k - j), \qquad j \le k. \tag{6.19}$$

Furthermore, (as in Section 6.1.1.1), let $g(k)$ be the corresponding holding and shortage cost rate at time $t + L$. We have

$$g(k) = -b_1 E(IL) + (h + b_1) E(IL^+) = -b_1 (k - \mu') + (h + b_1) \sum_{j=1}^{k} jP(IL = j). \tag{6.20}$$

As shown in Section 5.9.1, $g(k)$ is a convex function of the inventory position k. Furthermore $g(k) \to \infty$ as $|k| \to \infty$.

Define now

$$\overline{g}(y) = \sum_{j=1}^{Q} g(y + j),$$

where y is an integer. Clearly $\overline{g}(y)$ is also convex. Denote by R the finite integer y that minimizes $\overline{g}(y)$.

Lemma 6.1 Let x and z be integers. For a given z, $g(z + xQ)$ is convex in x. Let x_z be the unique integer so that $R + 1 \le z + x_z Q \le R + Q$. Then $g(z + xQ)$ is minimized with respect to x for $x = x_z$.

Proof It follows from the convexity of $g(k)$ that $g(z + xQ)$ is convex in x. Note that

$$g(z+(x+1)Q)-g(z+xQ)=\overline{g}(z+xQ)-\overline{g}(z+xQ-1). \quad (6.21)$$

Consider first any $x < x_z$. This means that $z + xQ \leq R$. Obviously $\overline{g}(z+xQ)-\overline{g}(z+xQ-1)\leq 0$. Similarly, $x > x_z$ implies $z + xQ > R + Q$ and $\overline{g}(z+xQ)-\overline{g}(z+xQ-1)\geq 0$. It follows that $g(z + xQ)$ is minimized with respect to x for $x = x_z$.

We are now ready to prove the following proposition.

Proposition 6.1 An (R, Q) policy is optimal.

Proof Consider any feasible policy. Let y_t be the inventory position at time t. The cost rate at time $t + L$ is then $g(y_t)$. From Lemma 6.1 we know that $g(y_t) \geq g(y_t')$ where $y_t' = y_t + nQ$ and n is the unique integer so that $y_t' \in \{R + 1, R + 2, \dots , R + Q\}$. Consequently, the long-run average cost must be greater than or equal to the long-run average value of $g(y_t')$. To determine these costs consider the stochastic process y_t'. Obviously y_t' is constant between the customer demands. Let D_t be a demand at some time t. Let y_t^- be the inventory position before the demand and y_t^+ the inventory position after the demand. Clearly

$$y_t^+ = y_t^- - D_t + mQ, \quad (6.22)$$

where m is nonnegative. Furthermore, due to our construction we must also have,

$$y_t'^+ = y_t'^- - D_t + m'Q, \quad (6.23)$$

where m' is an integer. Given $y_t'^-$ and D_t, the value of m' is unique because $y_t' \in \{R + 1, R + 2, \dots , R + Q\}$. The demand sizes are independent, so the different y_t' can be seen as a Markov chain with the finite state space $\{R + 1, R + 2, \dots , R + Q\}$. The steady state distribution can be shown to be uniform. (This can be done in essentially the same way as the proof of Proposition 5.1 in Section 5.3.1. We omit the details.)

The long-run average value of $g(y_t')$ is therefore $\overline{g}(R)/Q$. This is a lower bound on the long-run cost of any feasible policy. But this lower bound can be achieved by using an (R, Q) policy. Using an (R, Q) policy the inventory position is also uniform on $\{R + 1, R + 2, \ldots, R + Q\}$. (We obtain the costs by setting $A = 0$ in (6.4).) This completes the proof.

Proposition 6.1 can be generalized in different ways, for example to other cost structures and to periodic review. In the special case when $Q = 1$ the (R, Q) policy degenerates to an S policy with $S = R + 1$. This means that Proposition 6.1 also demonstrates the optimality of an S policy in case of no ordering costs and no constraints concerning the batch quantities.

For problems with continuous or Poisson demand, (R, Q) policies and (s, S) policies are equivalent. For such problems (s, S) policies are consequently also optimal.

6.2.2 *Optimality of (s, S) policies*

If we replace the fixed batch quantity in Section 6.2.1 by an ordering cost the optimal policy is under quite general conditions of the (s, S) type. This is more difficult to show, see e.g., Porteus (2002).

It is interesting to note, however, that (s, S) policies are not necessarily optimal for problems with service constraints. Consider, for example, a problem with discrete integral demand where s and S are integers. It may very well happen that no (s, S) policy provides a certain given service level exactly. The best (s, S) policy that satisfies the service constraint will consequently give a slightly higher service than what is required. In such a situation it may be possible to reduce the costs by varying the policy over time so that the average service level is exactly as prescribed.

Early optimality results were presented by Iglehart (1963) and Veinott (1966). More recent results are provided by Zheng (1991), Rosling (2002b), and Beyer and Sethi (1999).

6.3 Updating order quantities and reorder points in practice

In Chapters 2 - 5 we have presented different techniques for forecasting and determination of batch quantities and reorder points. We shall now illustrate how these techniques can be implemented in an inventory control system. We assume that we are dealing with a single-echelon system and independent items.

The forecasts are normally updated with a certain periodicity. In general, it is most practical to also update reorder points and batch quantities at these times, immediately after updating the forecasts. Let

t_F = forecast period.

We can think of the forecast period as, for example, one month. The time unit is not important. We can use one month as the time unit. In that case $t_F = 1$, but we can also express t_F in days ($t_F = 30$), or in years ($t_F = 1/12$). However, to avoid unnecessary errors it is recommended to use the same time unit in all inventory control computations.

Typically the forecasts are updated either by exponential smoothing (Section 2.4), or by exponential smoothing with trend (Section 2.5). It may also be reasonable to use a seasonal method (Section 2.6) for a few items. The forecasting method is, in general, chosen manually for each item. To specify the forecasting method, we also need the smoothing parameters that are part of the different forecasting methods. Usually it is practical to divide the items into a number of inventory control groups and let the forecasting technique as well as various inventory control parameters, like holding cost rate and service level, be identical for all items in the same group. See also Chapter 11.

When using exponential smoothing we update the average demand $\hat{a}_t$ at the end of each forecast period. If instead we use exponential smoothing with trend, we update both the average demand $\hat{a}_t$ and the trend $\hat{b}_t$. In either case we also update some error measure like MAD_t (Section 2.10).

In case of exponential smoothing the average demand per unit of time μ is obtained as

$$\mu = \hat{a}_t / t_F . \tag{6.24}$$

When using exponential smoothing with trend we are, in principle, assuming that the demand is increasing or decreasing linearly with time. Just after the forecast update the estimated demand in the coming period is $\hat{a}_t + \hat{b}_t$. A natural estimate of the demand rate in the middle of this period is then $\hat{a}_t / t_F + \hat{b}_t / t_F$. The trend is $\hat{b}_t / (t_F)^2$. The corresponding estimate of the demand rate in the beginning of the period, i.e., just after the update, is then $\hat{a}_t / t_F + \hat{b}_t / (2t_F)$. The estimated demand rate u time units after the update is $\hat{a}_t / t_F + \hat{b}_t / (2t_F) + u\hat{b}_t / (t_F)^2$.

The standard deviation of the demand per time unit is obtained according to (2.50) and (2.55) as

$$\sigma = \frac{1}{(t_F)^c}\sqrt{\frac{\pi}{2}}MAD_t, \tag{6.25}$$

where the parameter $c = 1/2$ if we assume that forecast errors in different time periods are independent. This can be regarded as the standard assumption. The parameter c is always in the interval (0.5, 1).

Assume that a continuous review (R, Q) policy is used for inventory control. Since the items are treated independently we shall consider a certain item with lead-time L. The lead-times are in general different for different items. Our first step is to update the batch quantity Q. The most common technique is to use the classical economic order quantity model and let the average demand per time unit μ replace the constant demand per unit of time. As in (4.3) we obtain

$$Q = \sqrt{\frac{2A\mu}{h}}. \tag{6.26}$$

The ordering cost, A, is in general, the same for items belonging to the same inventory control group. The holding cost, h, is usually determined as a certain percentage of the value of the item. This carrying charge should include capital costs as well as other types of holding costs. Usually the carrying charge is the same for all items in the same inventory control group, but the holding costs vary among the items because of different values of the items. A typical carrying charge could be something like 10 - 15 percent if we use one year as the time unit. The carrying charge is normally higher than the interest rate charged by the bank. See Sections 3.1.1 and 3.1.2.

Although (6.26) is intended for stationary demand, it is often also applied when using exponential smoothing with trend. In that case the average demand rate μ should correspond to the time when the batch is used. Consider a time interval of length τ starting at the time of the update. Let $D(\tau)$ be the stochastic demand during this interval and $g(\tau)$ the expected value of this demand. We have

$$g(\tau) = E\{D(\tau)\} = \int_0^{\tau}\frac{1}{t_F}(\hat{a}_t + \frac{\hat{b}_t}{2}) + \frac{\hat{b}_t}{t_F^2}u)du = (\hat{a}_t + \frac{\hat{b}_t}{2})\frac{\tau}{t_F} + \frac{\hat{b}_t}{2}\frac{\tau^2}{t_F^2}. \tag{6.27}$$

By setting $g(\tau)$ equal to a certain quantity d and solving for τ, we can estimate the time $\tau(d)$ until a certain quantity d has been demanded. We obtain $\tau(d)$ as the solution of a second order equation

$$\tau(d) = -t_F\left(\frac{\hat{a}_t}{\hat{b}_t} + \frac{1}{2}\right) + t_F\sqrt{\left(\frac{\hat{a}_t}{\hat{b}_t} + \frac{1}{2}\right)^2 + \frac{2d}{\hat{b}_t}}. \qquad (6.28)$$

We can use (6.27) and (6.28) for estimating the demand rate ahead of time in connection with determination of reorder point and batch quantity.

Consider, for example, an order just after the forecast update. Assume that the inventory position is equal to the reorder point R. What is then the average demand to be used in (6.26)? Let Q' be an estimate of the batch quantity, e.g., the previous batch size. We will start to consume the batch around time $\tau(R)$, and the whole batch will be consumed around time $\tau(R + Q')$. About half of the batch has been consumed at time $\tau' = \tau(R + Q'/2)$. A reasonable estimate of the average demand rate during the time when the batch is consumed is then

$$\mu = \frac{1}{t_F}(\hat{a}_t + \frac{\hat{b}_t}{2}) + \frac{\hat{b}_t}{t_F^2}\tau'), \qquad (6.29)$$

and we can use this μ instead of (6.24) in (6.26). Recall also from Section 4.1.2 that the costs are very insensitive to small errors in the batch quantity, so it may also be reasonable to use simpler approximations.

It is a little more complicated to take seasonal demand variations into account when determining batch quantities. In practice it is therefore quite common to disregard the effect of the seasonal variations on the batch quantities.

To be able to determine the reorder point, we next need to determine the distribution of the lead-time demand. Let μ' and σ' be the mean and average of the lead-time demand just after the forecast update. In case of exponential smoothing we have

$$\mu' = \frac{\hat{a}_t}{t_F}L, \qquad (6.30)$$

and in case of exponential smoothing with trend

$$\mu' = g(L)\,. \tag{6.31}$$

The standard deviation is obtained as

$$\sigma' = \sigma L^c = \sqrt{\frac{\pi}{2}}\, MAD_t \left(\frac{L}{t_F}\right)^c . \tag{6.32}$$

It is not common to take stochastic variations in the lead-time into account. One reason is that it is usually difficult to determine the lead-time distribution. If the lead-time variations are known and the deliveries are sequential it is easy to use the approximation based on (5.102) and (5.103).

Given the mean and standard deviation, the most common approach is to assume that the lead-time demand is normally distributed. For items with low demand it may sometimes be more appropriate to use a Poisson distribution or a compound Poisson distribution. If we assume that the normal distribution is used and that there is a given fill rate S_2 (See Section 5.4), it is easy to determine the reorder point R from (5.52). In case of compound Poisson demand we can apply (5.36) and (5.51).

Note that in this section we have used models for stationary demand also when there is a trend in demand. This approximation is usually satisfactory as long as the trend is relatively small compared to the average.

Example 6.3 We shall update forecast, batch quantity, and reorder point for an item controlled by a continuous review (R, Q) policy. The updates take place at the end of each month, and we use one month as our time unit, i.e., $t_F = 1$. We apply exponential smoothing with smoothing constant $\alpha = 0.1$ when updating both the forecast and MAD. At the end of the preceding month we obtained the forecast $\hat{a} = 132$ and $MAD = 42$. We have just received the demand during the last month as 92. The batch quantity is determined according to the classical economic order quantity model. The holding cost is \$1.5 per unit and month, and the ordering cost \$200. The lead-time is two months. The fill rate is required to be 95 percent. The demand can be regarded as continuous and normally distributed, and forecast errors during different time periods are assumed to be independent.

As our first step we update the forecast and MAD

$$\hat{a} = 0.9 \cdot 132 + 0.1 \cdot 92 = 128\,,$$

$$MAD = 0.9 \cdot 42 + 0.1 \cdot |132 - 92| = 41.8\,.$$

Next we determine the batch quantity

$$Q = \sqrt{\frac{2 \cdot 200 \cdot 128}{1.5}} = 184.75 \approx 185 .$$

It remains to determine the reorder point R from (5.52). First we obtain μ' and σ' as

$$\mu' = 2 \cdot 128 = 256 ,$$

$$\sigma' = \sqrt{2} \cdot \sqrt{\frac{\pi}{2}} \cdot 41.8 = 74.09 .$$

Finally, using the search procedure described in connection with (5.52) we obtain $R = 313.62 \approx 314$.

We have assumed a continuous review (R, Q) policy. Periodic review can be handled as described in Section 5.12. When using an (s, S) policy instead of an (R, Q) policy it is common in practice to first determine R and Q for an (R, Q) policy and then apply the simple approximation $s = R$ and $S - s = Q$.

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Problems

6.1* Consider Example 6.1 and the iterations in Table 6.1. What is the fill rate in the different iteration steps? Why?

6.2 Verify the transformation in Section 6.1.3.

6.3* Consider an item which is controlled by a continuous review (R, Q) policy. The forecast and *MAD* have just been updated by exponential smoothing as $\hat{a} = 100$ and MAD = 40. The forecast period is one month. The lead-time is two months. When adjusting the standard deviation to a different time, the constant c is set to 0.7. The lead-time demand is normally distributed.

a) Determine the reorder point for S_1 = 90 percent,
b) For this reorder point determine S_2 for Q = 25, Q = 100, and Q = 1200.

6.4* Simple exponential smoothing is used for updating the forecast each week. The smoothing constant is 0.2. *MAD* is also updated by exponential smoothing with smoothing constant 0.3. The demand during the past five weeks is given in the table. Before week 1 the forecast was 100 and *MAD* was 10.

Week	1	2	3	4	5
Demand	112	96	84	106	110

* Answer and/or hint in Appendix 1.

a) Update forecast and *MAD* for weeks 1-5. Determine the expected demand and variance for week 6.
b) Determine (after the update in period 5) batch quantity by the classical economic order quantity model and reorder point under the following assumptions:
Ordering cost: 100
Holding cost: 1 per unit and week
Lead-time: 2 weeks
$S_1 \geq 95\%$
Forecast errors in different periods are assumed to be independent.

6.5* The demand during the past five weeks is given.

Week	16	17	18	19	20
Demand	97	99	100	126	112

Forecasts are determined by both simple exponential smoothing and by exponential smoothing with trend. For simple exponential smoothing, the smoothing constant is 0.2. The same smoothing constant is used when updating the mean with the trend model. The smoothing constant for the trend is 0.4. When updating *MAD* the smoothing constant is 0.2. The forecast errors are assumed to be independent and normally distributed. Before the first week (16) the forecasted demand was 100.0 and the trend was assumed to be zero. *MAD* was 7.

a) Update the forecasts by both methods. Determine mean and standard deviation for week 21 after the update in week 20. Assume stationary stochastic demand. Use the forecast from simple exponential smoothing. Determine batch quantity by the classical economic lot size formula. Determine reorder point such that the fill rate is approximately 95%. Make the following assumptions:
Ordering cost: 2500
Holding cost: 10 per unit and week
Lead-time: 3 weeks
Continuous review
b) Determine S_1 for the chosen reorder point.

6.6 Consider a continuous review (R, Q) policy. The batch quantity $Q = 500$. The lead-time is two weeks. Both S_1 and S_2 must be at least 95%. Before week 1 the forecast was $\hat{x}_{0,1} = 100$ and $MAD_0 = 8$. Use simple exponential smoothing with $\alpha = 0.2$ (for both $\hat{a}$ and *MAD*) to update the forecast for weeks 1 to 5. The demands are:

week	1	2	3	4	5
d	113	101	108	105	95

Use the forecast from week 5 when determining the reorder point. The demand is normally distributed and deviations in different periods are independent.

7 COORDINATED ORDERING

In Chapters 3-6 it was assumed that different items in an inventory could be controlled independently. We shall now leave this assumption and consider situations where there is a need to coordinate orders for different items. In this chapter we shall still, as in Chapters 3-6, assume that the items are stocked at a single location. (Multi-stage inventory systems are dealt with in Chapters 8-10.) We consider traditional inventory costs and constraints, i.e., holding costs, ordering or setup costs, and backorder costs or service constraints.

When coordinating the replenishments for different items, it is common to use cyclic schedules, and especially so-called powers-of-two policies. In Section 7.1 we derive some important results for such policies.

There are two main reasons for coordinating the replenishments of a group of items. One reason, dealt with in Section 7.2, is that we wish to get a sufficiently smooth production load. Assume, for example, that a considered group of items is produced in the same production line. We then want to coordinate the orders for different items so that they are evenly spread over time.

The other main reason for coordinated replenishments, which is treated in Section 7.3, is completely opposite. We want to trigger orders for a group of items at the same time. This can be advantageous in many situations. It may be possible to get a discount if the total order from the same vendor is greater than a certain breakpoint. It may also be possible to reduce the transportation costs, for example, by filling a truckload. Sometimes the setup costs can also be lowered substantially if a group of similar items are produced together in a machine.

7.1 Powers-of-two policies

Both when using mathematical algorithms and when choosing schedules manually, it is very common to use so-called powers-of-two policies in connection with coordinated replenishments. This means that the cycle times are restricted to be powers of two times a certain basic period. If the basic period is, for example, one week, nonnegative powers of two give $2^0 = 1$ week, $2^1 = 2$ weeks, $2^2 = 4$ weeks, $2^3 = 8$ weeks etc. With negative powers of two we also obtain $2^{-1} = 1/2$ week, $2^{-2} = 1/4$ week, etc. A main advantage of such cycle times is that we obtain relatively simple cyclic schedules. Consider, for example, two items that are produced every fourth and every eighth week, respectively. The total cycle time is then eight weeks, since everything is repeated every eighth week. If instead of four and eight weeks we use the similar cycle times five and seven weeks, the total cycle time would be $5 \cdot 7 = 35$ weeks, i.e., more than four times longer. Assume, for example, that both items are produced in week 1. The item with cycle time five weeks is then produced in weeks 1, 6, 11, 16, 21, 26, 31, 36, etc., and the other item, with cycle time seven weeks, in weeks 1, 8, 15, 22, 29, 36, etc. So week 36 is the first time after week 1 when both items are produced.

Consider a number of items with constant continuous demand. Given holding costs and ordering costs we wish to determine suitable batch quantities, or equivalently cycle times. (See Section 4.1.1.) We shall show that a restriction to powers-of-two policies will give a solution which is very close to the optimal solution.

Consider first a single item. Recall the following result from Section 4.1.2.

$$\frac{C}{C^*} = \frac{1}{2}\left(\frac{Q}{Q^*} + \frac{Q^*}{Q}\right), \tag{7.1}$$

which gives the relative cost increase when deviating from the optimal batch quantity Q^* in the classical economic order quantity model. The expression (7.1) is valid also with a finite production rate (Section 4.2). Furthermore, since we are dealing with constant demand, d, we can just as well express the policy through the cycle time $T = Q/d$ where $T^* = Q^*/d$ is the optimal solution. This means that we can equivalently formulate (7.1) as

$$\frac{C}{C^*} = \frac{1}{2}\left(\frac{T}{T^*} + \frac{T^*}{T}\right). \tag{7.2}$$

We shall consider cycle times and the representation (7.2) when deriving our results on the approximation errors, but it is important to note that the results are also valid for the batch quantities.

Consider now a powers-of-two solution of a lot sizing problem, i.e., assume that the cycle time T has to be chosen as

$$T = 2^m q\,, \tag{7.3}$$

where m can be any integer and where, for the time being, we assume that the basic period q is given. Assume that T^* cannot be expressed according to (7.3). We then have to choose either the next lower or the next higher T satisfying (7.3). Due to (7.3), the ratio between these two values is 2. Note also that the best solution under the constraint (7.3) is not affected if q is multiplied by a power of two. If, for example, q is multiplied by 2, we can reduce m by 1 to get the same result.

What is the worst possible relative cost increase caused by restricting the solution with the constraint (7.3)? Because of the convexity, the worst possible error must occur when two consecutive values of m, say $m = k$ and $m = k + 1$ give the same error. Let $T < T^*$ correspond to $m = k$, and $2T > T^*$ to $m = k + 1$. We obtain

$$\frac{C}{C^*} = \frac{1}{2}\left(\frac{T}{T^*} + \frac{T^*}{T}\right) = \frac{1}{2}\left(\frac{2T}{T^*} + \frac{T^*}{2T}\right). \tag{7.4}$$

It is easy to see that (7.4) implies that

$$\frac{T^*}{T} = \frac{2T}{T^*} = \sqrt{2}\,, \tag{7.5}$$

and

$$\frac{C}{C^*} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) \approx 1.06\,. \tag{7.6}$$

We formulate this result as a proposition.

Proposition 7.1 For a given basic period q, the maximum relative cost increase of a powers-of-two policy is 6 percent.

We have only discussed a single item, but Proposition 7.1 is evidently also true if there are several items, because the worst case occurs when all items incur the maximum error of 6 percent.

Let us now assume that it is possible to change q. For a single item we will then get the optimal solution simply by choosing q equal to a power of two times T^*. If, however, we have N items (items $i = 1, 2, \dots, N$), we can, in general, not fit q perfectly to all cycle times T_i^*, which depend on the problem data for different items. The relative cost increase can be expressed as

$$\frac{C}{C^*} = \frac{\sum_{i=1}^{N} C_i}{\sum_{i=1}^{N} C_i^*} = \frac{\sum_{i=1}^{N} C_i^*(C_i / C_i^*)}{\sum_{i=1}^{N} C_i^*}. \tag{7.7}$$

We know from (7.2) and (7.5) that for a given q, each C_i / C_i^* can be expressed as

$$\frac{C_i}{C_i^*} = e(x_i) = \frac{1}{2}(2^{x_i} + 2^{-x_i}), \qquad -1/2 \le x_i \le 1/2, \tag{7.8}$$

i.e., $T_i / T_i^* = 2^{x_i}$ $(-1/2 \le x_i \le 1/2)$. The end points $x_i = -1/2$ and $x_i = 1/2$ correspond to the worst case (7.6). Let us now interpret the weights for the different values of x_i in (7.7), $C_i^* / \sum_{i=1}^{N} C_i^*$, as probabilities. Denote the corresponding distribution function on [-1/2, 1/2] by $F(x)$, i.e., $F(-1/2) = 0$ and $F(1/2) = 1$. We can see C/C^* in (7.7) as an expected value of $e(x)$ and reformulate (7.7) as

$$\frac{C}{C^*} = \int_{-1/2}^{1/2} e(x)dF(x). \tag{7.9}$$

Assume now that we change q by multiplying by 2^y, where $0 \le y \le 1$. (Recall that multiplying q by a power of two does not affect the solution, so we are considering the most general change.) If we do not change x this is equivalent to replacing x by $x + y$. However, if $x + y > 1/2$ it is advantageous to change x to $x - 1$. This means that a certain x is replaced by $x + y$ for $x + y \le 1/2$, and by $x + y - 1$ for $x + y > 1/2$. Consequently we have

$$\frac{C}{C^*}(y) = \int_{-1/2}^{1/2-y} e(x+y)dF(x) + \int_{1/2-y}^{1/2} e(x+y-1)dF(x)$$
$$= \int_{y-1/2}^{1/2} e(u)dF(u-y) + \int_{-1/2}^{y-1/2} e(u)dF(u-y+1). \tag{7.10}$$

For a given distribution $F(x)$ the minimum cost increase is obtained by minimizing (7.10) with respect to $0 \le y \le 1$.

We are now ready to prove Proposition 7.2, which shows that the maximum relative error is surprisingly low.

Proposition 7.2 If we can change the basic period q, the maximum relative cost increase of a powers-of-two policy is 2 percent.

Proof The average cost increase for $0 \le y \le 1$ must be at least as large as the minimum and we have from (7.10), by changing the order of integration,

$$\min_{0\le y\le 1} \frac{C}{C^*}(y) \le \int_0^1 \left(\int_{y-1/2}^{1/2} e(u)dF(u-y) + \int_{-1/2}^{y-1/2} e(u)dF(u-y+1) \right) dy$$

$$= \int_{-1/2}^{1/2} e(u) \left(\int_0^{u+1/2} dF(u-y) + \int_{u+1/2}^{1} dF(u-y+1) \right) du$$

$$= \int_{-1/2}^{1/2} e(u)\big(-F(-1/2) + F(u) - F(u) + F(1/2)\big)\, du = \int_{-1/2}^{1/2} e(u)du = \frac{1}{\sqrt{2}\ln 2} \approx 1.02.$$

The worst case will occur when the distribution $F(x)$ is uniform on $-1/2 \le x \le 1/2$, see (7.9). A change of q will then not make any difference. This completes the proof.

Powers-of-two policies are important ingredients of Roundy's so-called 98 percent approximation. See Sections 7.3.1.2 and 9.2.2.

7.2 Production smoothing

In general, it is a very complicated problem to control the stocks of different items in such a way that we get both low inventory costs and smooth capacity utilization. To simplify the problem it is common to first disregard stochastic demand variations and solve the remaining deterministic problem. Safety stocks are then usually determined in a second step using techniques for independent items, see Chapter 5. However, the deterministic problem can also be very difficult, especially if there are many items with time-varying demand and several capacity constraints.

When the demands for different items are relatively stable over an extended period, it is often advantageous to use cyclic schedules as a means to obtain a smooth production load. This means that each item is ordered periodically, and that the ordering periods for different items are chosen such that the load becomes as smooth as possible.

Consider, for example, a machine which is used for producing four items (items 1 - 4) all with about the same demand. It can then be a good idea to decide that item 1 is produced in weeks 1, 5, 9 ... , item 2 in weeks 2, 6, 10 ... , item 3 in weeks 3, 7, 11 ... , and item 4 in weeks 4, 8, 12 ... . This can be organized by applying periodic review order-up-to-S policies. Each item has a review period equal to four weeks, but the reviews take place in different weeks. Since the machine is producing exactly one of the items each week, the load will be very smooth. As we have discussed in Chapter 3, we need, for a given lead-time, more safety stock when applying a periodic review policy as compared to a policy with continuous review. But the alternative of using a continuous review policy would in this case probably yield large variations in the production load, which would in turn, result in long and uncertain lead-times. Longer lead-times would also mean that we need more safety stock. Furthermore, if we applied a continuous review policy, the capital tied up as work-in-process would increase and lead to additional holding costs. In a situation like this, the total costs would generally be much lower when using the cyclic schedules obtained through the periodic review policy.

In a general case with many items and several production facilities, it can be extremely difficult to find suitable cyclic schedules. In Section 7.2.1 we will deal with different mathematical approaches for solving the problem in the special case when there is only a single machine. In practice it is most common to choose cyclic schedules manually without using mathematical techniques.

When the demand is time-varying, one possible approach is to formulate the problem as a mixed integer program (MIP) and apply mathematical programming techniques to obtain a solution. A typical formulation involves a number of items with given demands over a planning horizon, and holding and setup costs as in the dynamic lot size problem (Section 4.5). To produce an item, we need to use certain production resources with limited capacities. We shall consider such a model in Section 7.2.2.

Situations where it is, for various practical reasons, very difficult to coordinate the replenishments for different items are quite common. It can then still be possible to get a reasonably smooth flow of orders simply by adjusting the order quantities. We discuss such adjustments in Section 7.2.3.

It is also common in practice to smooth production outside the inventory control system. Orders obtained from the inventory control system are then not automatically released to production. Release times and order quantities are instead adjusted with respect to the present production load. One possible planning rule can, for example, be to not release more orders than there is capacity to produce within some fixed time frame. Adjustments of the release times mean essentially that the safety stocks are allowed to vary over time in order to smooth the load. We know from the models in Chapters 4 and 5 that small changes in batch quantities and safety stocks will, in general, not affect the costs significantly, while larger changes can have a substantial impact. When adjusting the orders obtained from the inventory control system, it is therefore important to avoid large changes for individual items. In this context it is also interesting to note that ordering systems of KANBAN type will automatically limit the number of outstanding orders. See Sections 3.2.3.1 and 8.2.1.

7.2.1 The Economic Lot Scheduling Problem (ELSP)

7.2.1.1 Problem formulation

We shall now consider a problem that has been dealt with extensively in the inventory literature, the classical Economic Lot Scheduling Problem. This problem concerns the determination of cyclic schedules for a number of items with constant demands. Backorders are not allowed. The production rate is finite and we wish to minimize standard holding and ordering costs. Everything is as in the elementary model in Section 4.2 with one exception; the items are all produced in a single production facility, i.e., we have a capacity constraint. It turns out that this additional constraint transforms a simple model into a very challenging problem.

We assume that batch quantities and, equivalently, cycle times are kept constant over time. Let us introduce the following notation:

N = number of items,
h_i = holding cost per unit and time unit for item i,

A_i = ordering or setup cost for item i,
d_i = demand per time unit,
p_i = production rate ($p_i > d_i$),
s_i = setup time in the production facility for item i, independent of the sequence of the items,
T_i = cycle time for item i (the batch quantity $Q_i = T_i d_i$).

We shall also for simplicity define:

ρ_i = d_i / p_i,
τ_i = $\rho_i T_i$ = production time per batch for item i excluding setup time,
σ_i = $s_i + \tau_i$ = total production time per batch for item i.

Table 7.1 shows data for N = 10 items. This sample problem was first presented by Bomberger (1966) and has since then been used extensively in the literature on economic lot scheduling.

Table 7.1 Bomberger's problem (time unit = one day).

Item	1	2	3	4	5	6	7	8	9	10
$h_i \cdot 10^5$	0.2708	7.396	5.313	4.167	116.0	11.15	62.50	245.8	37.50	1.667
A_i	15	20	30	10	110	50	310	130	200	5
d_i	400	400	800	1600	80	80	24	340	340	400
p_i	30000	8000	9500	7500	2000	6000	2400	1300	2000	15000
s_i	0.125	0.125	0.25	0.125	0.50	0.25	1	0.5	0.75	0.125

7.2.1.2 The independent solution

Expressing the costs for item i, C_i, as a function of the cycle time, T_i, we have

$$C_i = \frac{A_i}{T_i} + h_i d_i (1 - \rho_i) \frac{T_i}{2}. \tag{7.11}$$

The problem is to minimize $\sum_{i=1}^{N} C_i$ subject to the constraint that all items should be produced in the common production facility.

A first approach to solving the problem could be to disregard the capacity constraint and simply optimize each item separately as in Section 4.2. The optimal cycle time when disregarding the capacity constraint is

$$T_i = \sqrt{\frac{2A_i}{h_i d_i (1-\rho_i)}}, \tag{7.12}$$

and the corresponding cost

$$C_i = \sqrt{2A_i h_i d_i (1-\rho_i)}. \tag{7.13}$$

Note that (7.11) and (7.12) are equivalent to (4.6) and (4.7) if we replace T_i by Q_i/d_i.

Table 7.2 shows this "independent" solution of Bomberger's problem. The last row gives the corresponding production times per lot for each of the 10 items.

Table 7.2 Independent solution of Bomberger's problem.

Item	1	2	3	4	5	6	7	8	9	10
T_i	167.5	37.7	39.3	19.5	49.7	106.6	204.3	20.5	61.5	39.3
C_i	0.179	1.060	1.528	1.024	4.428	0.938	3.034	12.668	6.506	0.255
σ_i	2.36	2.01	3.56	4.29	2.49	1.67	3.04	5.87	11.20	1.17

The sum of the costs, $\underline{C} = \sum_{i=1}^{10} C_i = 31.62$, is evidently a lower bound for the total costs, since we have disregarded the capacity constraint.

Is the solution feasible? No, it is relatively easy to see that the solution in Table 7.2 cannot be implemented. Consider, for example, items 4, 8, and 9. Assume that the production of item 9 starts at some time t. The production of the following batch will then start at time t + 61.5 etc. Since $\sigma_9 = 11.20$, the production facility is occupied by item 9 during the interval (t, t + 11.20). Items 4 and 8 have cycle times 19.5 and 20.5. This means that we must be able to produce one batch of item 4 and one batch of item 8 in the interval (t, t + 20.5). Given the production schedule for item 9, this implies that both items must be produced in (t + 11.20, t + 20.5). The length of this interval is only 9.30, while $\sigma_4 + \sigma_8 = 10.16$. The independent solution can consequently not be implemented.

7.2.1.3 Common cycle time

Does a feasible solution exist? Note first that if at least one setup time is positive, an obvious *necessary* condition for a feasible solution to exist is that

$$\sum_{i=1}^{N} \rho_i < 1 . \tag{7.14}$$

The left-hand side of (7.14) is the ratio of the time the production facility must be busy (excluding setup time). Since we also need some time for the setups, we can see that (7.14) is necessary. (If all setup times are zero, (7.14) should be replaced by $\sum_{i=1}^{N} \rho_i \leq 1$.) It is easy to verify that (7.14) is satisfied for Bomberger's problem.

It turns out though, that the condition (7.14) is also *sufficient* for feasibility. If it is satisfied, it is always possible to find a feasible solution where all items have a *common cycle time*. Denote the common cycle time by T. During the cycle time we produce all items, each time in the same order. The production quantity for an item is the demand during the cycle time. By choosing T sufficiently large, we can reduce the ratio of the time needed for setups as much as we need. This explains why (7.14) is sufficient.

Given the assumption of a common cycle time, the problem now is to minimize

$$C = \sum_{i=1}^{N} \left(\frac{A_i}{T} + h_i d_i (1-\rho_i) \frac{T}{2} \right), \tag{7.15}$$

with respect to the constraint that the common cycle time must be able to accommodate production lots of all items

$$\sum_{i=1}^{N} \sigma_i = \sum_{i=1}^{N} (s_i + \rho_i T) \leq T . \tag{7.16}$$

Note that the constraint (7.16) can also be expressed as

$$T \geq \frac{\sum_{i=1}^{N} s_i}{1 - \sum_{i=1}^{N} \rho_i} = T_{min} , \tag{7.17}$$

i.e., it simply means a lower bound for the cycle time.

If we disregard (7.17), the optimization of (7.15) is similar to (7.12) and we obtain

$$\hat{T} = \sqrt{\frac{2\sum_{i=1}^{N} A_i}{\sum_{i=1}^{N} h_i d_i (1-\rho_i)}} . \tag{7.18}$$

Since (7.15) is convex in T, the optimal solution, T_{opt}, is obtained as

$$T_{opt} = \max(\hat{T}, T_{min}), \tag{7.19}$$

i.e., if $\hat{T} < T_{min}$ the best we can do is to choose $T_{opt} = T_{min}$.

For Bomberger's problem we obtain $\hat{T} = 42.75$ and $T_{min} = 31.86$, and consequently, $T_{opt} = \hat{T} = 42.75$. This gives the costs $\overline{C} = 41.17$, which is an upper bound for the optimal solution since we have enforced the additional constraint that all cycle times have to be equal.

At this stage we know that the optimal costs, C^*, are in the interval $\underline{C} = 31.62 \leq C^* \leq \overline{C} = 41.17$. For Bomberger's problem it is rather evident that the upper bound is not especially tight. If we look at the independent solution in Table 7.2, we can see that some of the cycle times are very different, and we can therefore not expect a good solution with a common cycle time. For problems where the individual cycle times are reasonably similar, we can, on the other hand, expect the common cycle approach to give a very good approximation.

We shall now consider two approaches for deriving better solutions.

7.2.1.4 Bomberger's approach

Bomberger (1966) generated feasible upper bound solutions by a relatively simple dynamic programming model. First it is assumed that each cycle time T_i is an integer multiple of a basic period W, i.e., $T_i = n_i W$ for some positive integer n_i. Bomberger also makes the very restrictive assumption that W should be able to accommodate production of all items. To see that this condition is not necessary, consider two items (item 1 and item 2), which both have $T_1 = T_2 = 2W$. We can then produce item 1 in periods 1, 3, 5 ... , and item 2 in periods 2, 4, 6 ... , and we will never have to produce both items in the same basic period. The condition will, however, obviously guarantee that the obtained solution is feasible. Let

$F_i(w)$ = minimum cost of producing items $i + 1$, $i + 2$, ... , N when the available capacity in the basic period is w, i.e., $W - w$ has been used for items 1, 2, ... , i.

We now have

$$F_{i-1}(w) = \min_{n_i} \{C_i(n_i W) + F_i(w - \sigma_i)\}, \tag{7.20}$$

where $C_i(n_i W)$ are the costs (7.11) for item i with $T_i = n_i W$, $\sigma_i = s_i + \rho_i n_i W$, and the integer n_i is subject to the constraint

$$1 \leq n_i \leq (w - s_i) / \rho_i W. \tag{7.21}$$

Note that the upper bound in (7.21) is equivalent to $\sigma_i \leq w$.

It is obvious that $F_N(w) = 0$ for all $w \geq 0$. In the first step we determine $F_{N-1}(w)$ for a suitable grid of values of $w \geq 0$ from (7.20). Given $F_{N-1}(w)$, we can next determine $F_{N-2}(w)$, etc. $F_0(W)$ gives the minimum costs when the basic period is equal to W. In the final step we need to optimize $F_0(W)$ with respect to W. Bomberger's solution of his example gave the costs $C = 36.65$ for $W = 40$, $n_i = 1$ for $i \neq 7$, and $n_7 = 3$. This solution is a considerable improvement as compared to the common cycle approach.

Bomberger's approach is simple and will always provide a feasible solution (if a feasible solution exists). Still, the assumption that W should be able to accommodate production of all items is very restrictive. Later contributions (Elmaghraby, 1978, and Axsäter, 1984, 1987), have improved the approach and provide considerably better solutions.

7.2.1.5 A simple heuristic

We shall now consider a completely different heuristic technique (essentially according to Doll and Whybark, 1973). The procedure means that we successively improve the multipliers n_i and the basic period W according to the following iterative procedure:

1. Determine the independent solution and use the shortest cycle time as the initial basic period W.

2. Given W, choose powers-of-two multipliers, ($n_i = 2^m$, $m \geq 0$), to minimize the item costs (7.11).

3. Given the multipliers n_i, minimize the total costs

$$C = \sum_{i=1}^{N} \left(\frac{A_i / n_i}{W} + h_i d_i (1-\rho_i) n_i \frac{W}{2} \right),$$

with respect to W. We obtain

$$W = \sqrt{\frac{2\sum_{i=1}^{N} A_i / n_i}{\sum_{i=1}^{N} h_i d_i (1-\rho_i) n_i}}.$$

4. Go back to Step 2 unless the procedure has converged. In that case, check whether the obtained solution is feasible. If the solution is infeasible, try to adjust the multipliers and then go back to Step 3.

The main disadvantage of the considered heuristic is that there is no guarantee for even a feasible solution. On the other hand, the computations are very easy.

We shall apply the heuristic to Bomberger's problem. In Step 1 we start with the shortest cycle time in Table 7.2 as our basic period W = 19.5. In Step 2 we obtain the powers-of-two multipliers $n_1 = 8$, $n_2 = 2$, $n_3 = 2$, $n_4 = 1$, $n_5 = 2$, $n_6 = 4$, $n_7 = 8$, $n_8 = 1$, $n_9 = 4$, and $n_{10} = 2$. In Step 3 we get $W = 20.30$. At this stage the algorithm has converged and we have to check whether the solution is feasible. We can see that this is not the case by using an argument which is very similar to the argument that we used to show that the independent solution is infeasible. We consider again items 4, 8, and 9, and determine $\sigma_4 = 4.45$, $\sigma_8 = 5.81$, and $\sigma_9 = 14.55$. Consider a basic period when item 9 is produced. Since items 4 and 8 are produced in all basic periods, the production of all three items must take place in the considered basic period, i.e., during the time 20.30. This is obviously impossible. A major problem is the long production time for item 9. To reduce σ_9, we change n_9 from 4 to 2 and start the iterations in Step 3. We get $W = 23.42$ and no more changes of the multipliers, see Table 7.3.

It turns out that this solution is feasible, see Table 7.4. The total production time is below $W = 23.42$ in each basic period. The total costs are C = 32.07, i.e., very close to the lower bound. This is also the best known solution of the problem.

Table 7.3 Solution of Bomberger's problem with $W = 23.42$.

Item	1	2	3	4	5	6	7	8	9	10
n_i	8	2	2	1	2	4	8	1	2	2
σ_i	2.62	2.47	4.19	5.12	2.37	1.50	2.87	6.63	8.71	1.37

Table 7.4 Feasible production plan.

Basic period	Items	Production time
1	4, 8, 2, 9	22.93
2	4, 8, 3, 5, 10, 1	22.30
3	4, 8, 2, 9	22.93
4	4, 8, 3, 5, 10, 6	21.18
5	4, 8, 2, 9	22.93
6	4, 8, 3, 5, 10, 7	22.55
7	4, 8, 2, 9	22.93
8	4, 8, 3, 5, 10, 6	21.18

The solution in Table 7.4 is repeated every 8th period. Note that it is easier to check feasibility when using powers-of-two policies since the total cycle time is usually relatively short. There are still, however, a large number of plans that correspond to the solution in Table 7.3. If $n_i = 4$, for example, we can produce item i in periods 1 and 5, or in periods 2 and 6, or in periods 3 and 7, or in periods 4 and 8, i.e., there are 4 possibilities. More generally the number of possibilities for item i is equal to n_i. We can, however, without any lack of generality, always allocate one of the items with maximum n_i, say item 1, in some arbitrary period. If, consequently, we disregard the allocation of item 1, the number of remaining possibilities can be obtained as the product of the multipliers of the remaining items 2 - 10, i.e., $2 \cdot 2 \cdot 1 \cdot 2 \cdot 4 \cdot 8 \cdot 1 \cdot 2 \cdot 2 = 1024$.

Although the classical economic lot scheduling problem involves only a single production facility, it can also be of interest in more general situations. For example, if there are several production facilities with limited capacities, it is quite common that one of the constraints constitutes the real bottleneck. It is then a reasonable approach to first derive a plan that only takes this constraint into account, and then in a second step, try to adapt the plan to other capacity limitations.

A more detailed overview of different approaches to solving the problem is provided in Elmaghraby (1978). Feasibility issues are analyzed by Hsu (1983).

7.2.1.6 Other problem formulations

In Sections 7.2.1.1 - 7.2.1.5 we have considered the classical Economic Lot Scheduling Problem. Several papers have dealt with a variation of this problem. This more general formulation of the problem allows the lot sizes to vary over time. There is still a cycle time T, which is the overall period of the system. The schedule repeats itself every T units of time. Each item is produced during T but some items may be produced more than once. Furthermore, the batch sizes of these runs may be different. See, for example, Dobson (1987), Roundy (1989), and Zipkin (1991).

Gallego and Roundy (1992) allow backorders, while Gallego and Moon (1992) consider a situation where setup times can be exchanged by setup costs.

There are also quite a few papers dealing with stochastic demand. In case of stochastic demand it is necessary to allow backorders and/or capacity variations, for example by using overtime production. See Sox et al. (1999) for a review. They classify the existing research approaches in two categories: cyclic sequencing and dynamic sequencing. The cyclic sequencing category uses a fixed cyclic schedule on the production facility, while the lot sizes are varied to meet demand variations. The cyclic schedule can, for example, be obtained from a deterministic model. Dynamic sequencing means that both the production sequence and the lot sizes are varied. Examples of the cyclic sequencing approach are Gallego (1990), Bowman and Muckstadt (1993, 1995), and Federgruen and Katalan (1996a, b). Papers considering dynamic sequencing are e.g., Graves (1980), and Sox and Muckstadt (1997).

7.2.2 Time-varying demand

7.2.2.1 A generalization of the classical dynamic lot size problem

Recall the classical dynamic lot size problem in Section 4.5. In this section we shall consider a generalization of this problem. Instead of a single item there are N items. Furthermore, these items are produced in the same machine, which has limited capacity. (The generalization to several machines is relatively straightforward.) For simplicity, it is assumed (as in Section 4.5) that all events take place in the beginning of a period. A quantity that is produced in period t can also be delivered to customers in period t. The demands for different items are given but vary over time. No backorders are allowed. Let us introduce the following notation:

N = number of items,
T = number of periods,
$d_{i,t}$ = demand for item i in period t,
a_i = setup time for item i,
b_i = operation time per unit for item i,
q_t = available time in the machine in period t,
$M_{i,t}$ = upper bound for the production of item i in period t,
A_i = ordering or setup cost for item i,
h_i = holding cost per unit and time unit for item i,
$x_{i,t}$ = production quantity of item i in period t,
$y_{i,t}$ = inventory of item i after the demand in period t, $y_{i,0} = 0$,
$\delta_{i,t} = \begin{cases} 1 \text{ if } x_{i,t} > 0, \\ 0 \text{ otherwise,} \end{cases}$
C = total variable costs.

We wish to choose production quantities in different periods so that the sum of the ordering and holding costs C are minimized. The considered problem can be formulated as a Mixed Integer Program (MIP).

$$C = \min \sum_{i=1}^{N} \left(A_i \sum_{t=1}^{T} \delta_{i,t} + h_i \sum_{t=1}^{T} y_{i,t} \right), \tag{7.22}$$

subject to

$$\sum_{i=1}^{N} \left(a_i \delta_{i,t} + b_i x_{i,t} \right) \le q_t, \quad t = 1,2,....,T, \tag{7.23}$$

$$y_{i,t} = y_{i,t-1} + x_{i,t} - d_{i,t}, \quad t = 1,2,....,T, i = 1,2,...,N, \tag{7.24}$$

$$x_{i,t} - M_{i,t}\delta_{i,t} \le 0, \quad t = 1,2,....,T, i = 1,2,...,N, \tag{7.25}$$

$$x_{i,t} \ge 0, \quad y_{i,t} \ge 0, \quad \delta_{i,t} = 0 \text{ or } 1, \quad t = 1,2,....,T, i = 1,2,...,N. \tag{7.26}$$

We need the constraint (7.25) to enforce that production in a period implies that there is a corresponding setup time and setup cost. If there is no upper bound on $x_{i,t}$, we can let $M_{i,t}$ be very large. In the sequel it is assumed that this is the case.

Although the considered problem can be seen as a minor variation of the classical dynamic lot size problem, the model is quite complex. If, for example $N = 100$ and $T = 12$, there are 1200 integer variables and 2400 nonnegative continuous variables. The number of constraints (7.23) - (7.25) is 2412.

A possible approach is to eliminate the capacity constraints (7.23) by a Lagrangian relaxation. Let $\lambda_t \geq 0$ be the multiplier for period t. We get the Lagrangean:

$$L = -\sum_{t=1}^{T} \lambda_t q_t + \min \sum_{i=1}^{N} \sum_{t=1}^{T} \left((A_i + a_i \lambda_t) \delta_{i,t} + h_i y_{i,t} + b_i x_{i,t} \lambda_t \right). \quad (7.27)$$

It is easy to see that the following proposition is true (Problem 7.7).

Proposition 7.3 For any nonnegative multipliers $L \leq C$.

Consider (7.27) together with the constraints (7.24) - (7.26). Because we have eliminated the capacity constraint (7.23) we can determine L by optimizing each item separately for certain given multipliers. For item i, the problem that we need to solve is a generalized dynamic lot size problem (see Section 4.5) with a time-variable setup cost $A_i + a_i\lambda_t$ and an additional time-variable production cost per unit $b_i \lambda_t$. Also this more general problem is easy to solve, e.g., by dynamic programming.

If we want to get a tight lower bound for C we can solve the *dual* problem.

$$D = \max_{\lambda_1, \lambda_2, \ldots, \lambda_T} L. \quad (7.28)$$

In general, we will get a duality gap i.e., $D < C$ because our MIP is nonconvex. A tight lower bound is very useful, though. We can, for example, check whether a heuristic feasible solution to the original problem gives a cost that is reasonably close to the lower bound. In that case we know that the approximate solution is acceptable. Furthermore, if we want to solve the original problem by a branch-and-bound procedure, we can determine the needed associated lower bounds from the dual problem.

Let us now consider another interesting approach for solving the problem (7.22) - (7.26). Note first that in an optimal solution of the considered problem all items must have $y_{i,T} = 0$. Otherwise we would just get unnecessary holding costs. Let us now focus on one of the items. (For simplicity, we suppress index i.) We shall say that a production plan, $(x_1, x_2, \dots, x_T)$, is *demand-feasible* if all demands are satisfied without delays and the end stock is zero. It is easy to verify that the set of demand-feasible plans is convex, i.e., if two plans are demand-feasible, a convex combination of these plans is also demand-feasible (Problem 7.8).

In Section 4.5 we showed that the optimal solution of the dynamic lot size problem must satisfy *Property 1*, meaning that *the production in a period must cover the demand in an integer number of consecutive periods*. This is no longer true for our more general problem with a capacity constraint. However, it turns out that the set of such plans constitute the extreme points of the convex set of all demand-feasible plans.

Proposition 7.4 The set of plans satisfying *Property 1* constitute the extreme points of all demand-feasible plans.

Proof Consider first a demand-feasible plan, which does not satisfy *Property 1*. Then there must exist a period j such that the demand in this period is covered partly by the production in some period $k < j$, and partly by the production in period j. We denote this plan by x. Let ε be a small number and consider two other plans x' and x'', which are almost identical to x. The only differences are that in x' we replace x_k by $x_k + \varepsilon$ and x_j by $x_j - \varepsilon$, and in x'' we replace x_k by $x_k - \varepsilon$ and x_j by $x_j + \varepsilon$. Clearly these plans are demand-feasible provided ε is sufficiently small. But $x = (x' + x'')/2$, so x is not an extreme point.

Consider then the set of plans satisfying *Property 1*. Let x be such a plan and assume that it can be expressed as a convex combination of two other such plans y and z. We shall show that this assumption leads to a contradiction. It follows that all periods without production in x must be without production also in y and z. This means that the difference between y (or z) and x is that some of the batches associated with x have been aggregated. (Recall that we by assumption cannot have the same batches.) Any given holding cost $h > 0$ will then give higher holding costs for y and z than for x. It is easy to see that this must then also be true for a convex combination of y and z. This is a contradiction and completes the proof.

Proposition 7.4 implies that any demand-feasible plan can be expressed as a convex combination of plans satisfying *Property 1*. Let us now reintroduce index i for item i. Consider the plans for item i satisfying *Property 1* and assume that these plans are numbered in some way. Let R_i be the total number of plans. ($R_i \leq 2^{T-1}$, see Problem 7.9.) Furthermore, let

$x_{i,t,r}$ = production of item i in period t when using plan r,

$c_{i,r}$ = costs for item i when using plan r,
$\beta_{i,t,r}$ = capacity requirements for item i in period t when using plan r.

Using Proposition 7.4 we can express the production of item i in period t for any demand-feasible plan as

$$x_{i,t} = \sum_{r=1}^{R_i} x_{i,t,r}\theta_{i,r} , \tag{7.29}$$

where

$$\sum_{r=1}^{R_i} \theta_{i,r} = 1 , \tag{7.30}$$

and $\theta_{i,r} \geq 0$.

We are now ready to formulate the following linear programming model:

$$\min C = \sum_{i=1}^{N} \sum_{r=1}^{R_i} c_{i,r}\theta_{i,r} , \tag{7.31}$$

$$\sum_{i=1}^{N} \sum_{r=1}^{R_i} \beta_{i,t,r}\theta_{i,r} \leq q_t , \quad t = 1,2,...,T, \tag{7.32}$$

$$\sum_{r=1}^{R_i} \theta_{i,r} = 1, \quad i = 1,2,...,N, \tag{7.33}$$

$$\theta_{i,r} \geq 0, \quad i = 1,2,...,N, r = 1,2,...,R_i . \tag{7.34}$$

Although (7.29) is exact, the considered linear program is an approximation unless all $\theta_{i,r}$ are 0 or 1. Assume, for example, that for item i we have $\theta_{i,1} = \theta_{i,2} = 1/2$, i.e., we are using a *mixed* strategy with weights 1/2 for plans 1 and 2. Assume also that plan 1 has production in period t but not plan 2. According to (7.31) the setup cost in period t is only half of the correct setup cost. In the same way we will underestimate the setup time in (7.32).

Although, the linear program (7.31) - (7.34) is approximate, it turns out that the approximation in many important cases is very good. If we solve the linear program we will get a solution that has at most $T + N$ nonzero variables, i.e., the number of constraints (excluding nonnegativity constraints).

Assume that there are m items that have more than one positive $\theta_{i,r}$. The total number of nonzero variables is then less or equal to $2m + N - m = m + N \leq T + N$ so we get $m \leq T$. If there are many items and few time periods the fraction of mixed strategies will be very small. Let, for example, $N = 500$ and $T = 5$. We then have $m \leq 5$. This means that the fraction of items with mixed strategies is at most 1 percent. The resulting solution may not be optimal and may also be infeasible. Still, from a practical point of view, the solution is probably very good. The real capacity constraint is usually not that rigid. It may, for example, be possible to use some overtime. The resulting costs will be very close to the optimal costs. Note, however, that in case of a small N and a large T the considered model is of less interest. See, e.g., Example 7.1 below.

The first model of this type was formulated by Manne (1958).

Example 7.1 Consider a problem with two items and three periods. See Table 7.5. There are no initial stocks. The available production capacity is 100 time units per period. For both items the setup time is 15 time units and the operation time is one unit of time per unit. The holding cost is 1 per unit and time unit. There is no setup cost.

Table 7.5 Demands in different periods.

Item	Demand, period 1	Demand, period 2	Demand, period 3
1	25	25	75
2	20	50	25

First we determine the extreme points of the set of demand-feasible plans, i.e., the plans satisfying *Property 1*. See Table 7.6.

Table 7.6 Demand-feasible plans.

Item	Plan	Production, period 1	Production, period 2	Production, period 3
1	1	125	0	0
1	2	50	0	75
1	3	25	100	0
1	4	25	25	75
2	1	95	0	0
2	2	70	0	25
2	3	20	75	0
2	4	20	50	25

The holding costs (no setup costs) and the capacity requirements for the demand-feasible plans are:

$c_{1,1} = 175$	$c_{1,2} = 25$	$c_{1,3} = 75$	$c_{1,4} = 0$
$c_{2,1} = 100$	$c_{2,2} = 50$	$c_{2,3} = 25$	$c_{2,4} = 0$
$\beta_{1,1,1} = 140$	$\beta_{1,1,2} = 65$	$\beta_{1,1,3} = 40$	$\beta_{1,1,4} = 40$
$\beta_{2,1,1} = 110$	$\beta_{2,1,2} = 85$	$\beta_{2,1,3} = 35$	$\beta_{2,1,4} = 35$
$\beta_{1,2,1} = 0$	$\beta_{1,2,2} = 0$	$\beta_{1,2,3} = 115$	$\beta_{1,2,4} = 40$
$\beta_{2,2,1} = 0$	$\beta_{2,2,2} = 0$	$\beta_{2,2,3} = 90$	$\beta_{2,2,4} = 65$
$\beta_{1,3,1} = 0$	$\beta_{1,3,2} = 90$	$\beta_{1,3,3} = 0$	$\beta_{1,3,4} = 90$
$\beta_{2,3,1} = 0$	$\beta_{2,3,2} = 40$	$\beta_{2,3,3} = 0$	$\beta_{2,3,4} = 40$

The linear program (7.31) - (7.34) is then obtained as:

$$\min C = 175\theta_{1,1} + 25\theta_{1,2} + 75\theta_{1,3} + 100\theta_{2,1} + 50\theta_{2,2} + 25\theta_{2,3}, \quad (7.35)$$

$$\begin{aligned} &140\theta_{1,1} + 65\theta_{1,2} + 40\theta_{1,3} + 40\theta_{1,4} \\ &+ 110\theta_{2,1} + 85\theta_{2,2} + 35\theta_{2,3} + 35\theta_{2,4} \le 100, \end{aligned} \quad (7.36)$$

$$115\theta_{1,3} + 40\theta_{1,4} + 90\theta_{2,3} + 65\theta_{2,4} \le 100, \quad (7.37)$$

$$90\theta_{1,2} + 90\theta_{1,4} + 40\theta_{2,2} + 40\theta_{2,4} \le 100, \quad (7.38)$$

$$\theta_{1,1} + \theta_{1,2} + \theta_{1,3} + \theta_{1,4} = 1, \quad (7.39)$$

$$\theta_{2,1} + \theta_{2,2} + \theta_{2,3} + \theta_{2,4} = 1, \quad (7.40)$$

$$\theta_{1,1}, \theta_{1,2}, \theta_{1,3}, \theta_{1,4}, \theta_{2,1}, \theta_{2,2}, \theta_{2,3}, \theta_{2,4} \ge 0. \quad (7.41)$$

Let m be the number of items that have more than one positive $\theta_{i,r}$. Our bound $m \le T = 3$ is in this case of no interest, because there are only two items.

When solving the linear program (7.35) - (7.41) we get the following nonnegative $\theta_{i,r}$: $\theta_{1,2} = 3/4$, $\theta_{1,4} = 1/4$, and $\theta_{2,3} = 1$. The corresponding production plans are (175/4, 25/4, 75) for item 1 and (20, 75, 0) for item 2. The optimal cost obtained from the linear program is 175/4. It is easy to see that the obtained solution is not feasible (Problem 7.10).

7.2.2.2 Application of mathematical programming approaches

Recently there has been a renewed interest in mathematical programming models for production-inventory planning. One reason is that such models are quite often included as so-called Advanced Planning Systems (APS) in modern Enterprise Resource Planning (ERP) systems. An overview of APS systems is given by Fleischmann and Meyr (2003). ERP systems are discussed in Section 8.2.4.

As illustrated in Section 7.2.2.1, detailed models dealing with individual items are, in general, quite complex due to nonlinear setup times and costs. Still, it is possible to solve relatively large problems of this kind quite efficiently. Shapiro (1993) gives an overview of these types of models. See also Billington et al. (1983) and Eppen and Martin (1987).

One possibility to partly avoid the difficulties associated with setup times and setup costs is to use a *hierarchical planning* procedure. This means that the planning is carried out at two (or more) hierarchical levels. This is in line with standard industrial planning procedures. At an upper level an *aggregate* problem is considered. This problem concerns aggregate entities like product groups and machine groups. At the lower level the aggregate plan is *disaggregated* into a detailed plan for individual items and machines. The disaggregation is usually carried out by a simple heuristic procedure. In a model representing the aggregate level, it is usually necessary and reasonable to disregard the nonlinearities because they are associated with individual items. It is consequently possible to use linear programming. Furthermore, the number of variables and constraints are reduced by considering product groups and machine groups instead of individual items and machines. Therefore the model becomes less complex.

An overview of hierarchical planning models is given by Bitran and Tirupati (1993). See also the discussion of different planning concepts in De Kok and Fransoo (2003).

7.2.3 Production smoothing and batch quantities

In many situations it is not practical to smooth production by active coordination of the replenishments. If, for example, the demands of different items are varying substantially over time, it is difficult to smooth production by using cyclic schedules. It may still be an important goal to avoid excessive queues in production, though. A remaining possibility is to try to adjust the batch quantities in order to obtain a reasonably smooth load.

Many companies have found that smaller order quantities may smooth the production load and reduce the queues in production. There is a simple explanation for this. If the orders arriving to production can be seen as a stochastic process, smaller batches will reduce the variations over time due to the laws of large numbers. The load during a certain time is then built up of a larger number of smaller batches. If the orders are more or less independent, this will clearly smooth the load. On the other hand, we cannot use order quantities that are too small, since this will mean that the setup times will take too much of the available capacity into account and increase the queues. As usual, we have to find a suitable middle way.

The models which we have dealt with in Chapters 4 - 6 do not address these types of questions. In these models it is assumed that the lead-time is constant, or that the lead-time distribution is given but independent of the batch quantity. But from our discussion it is obvious that there are situations when this is not true and the lot sizes strongly affect the lead-time distribution. There are some lot-sizing models that explicitly take the queuing situation in production into account. We shall consider a simple such model essentially according to Karmarkar (1987, 1993). See also Axsäter (1980, 1986), Bertrand (1985), and Zipkin (1986).

Consider a machine in a large multi-center shop. This machine is used for producing a number of similar items having the same batch size. We define

d = average total demand for all items, units per time unit,
p = average processing rate, units per time unit,
Q = batch quantity for an item,
s = setup time for a batch,
T = average time in the system for a batch.

We shall, for simplicity, assume that the batches arrive at the machine as a Poisson process with rate $\lambda = d/Q$. This is a reasonable approximation if there are relatively many items. Let us also, for simplicity, think of the processing time as exponentially distributed. The average processing time for a batch is $1/\kappa = s + Q/p$ and the service factor $\rho = \lambda/\kappa = ds/Q + d/p$. The average time in the considered (M/M/1) queuing system is then

$$T = \frac{1/\kappa}{1-\rho} = \frac{s + Q/p}{1 - ds/Q - d/p}. \tag{7.42}$$

Note that we must require $\rho < 1$, or equivalently, $Q > ds/(1 - d/p)$, i.e., the cycle time must on average accommodate both the setup time and the operation time.

By minimizing the average time in the system, T, with respect to Q, it is possible to show that the minimum occurs for

$$Q = \frac{ds}{1 - d/p} + \frac{s\sqrt{dp}}{1 - d/p}. \tag{7.43}$$

The average time in the system is large both when using batch quantities that are too small and when using batch quantities that are too large. When the batch quantities are too small the setup times will require too much capacity. When the batch quantities are too large the production time for a batch is long and there are also more stochastic variations in the production load, which will cause longer queuing times.

If we wish to minimize the sum of holding costs for work-in-process we should, in the considered case, use Q according to (7.43). However, there are also other objectives when choosing the batch size. For example, there are holding costs for stocks of completed items. There may also be ordering costs. The holding costs for completed items are clearly affected by the replenishment lead-time. We can see (7.42) as the lead-time (or part of the lead-time) when modeling the stocks of completed items. (This means that we disregard that the lead-time is stochastic.) Doing so, we can include holding costs for both work-in-process and the stocks of completed items in the same model when determining batch quantities and reorder points.

Note that the average time in the system for a batch is, in steady state, not affected by the reorder points for completed items.

7.3 Joint replenishments

In Section 7.2 we have considered situations where we want to have orders for different items spread out over time in order to get a smooth production load. In this section we shall deal with the opposite problem, i.e., we shall consider a group of items which should be replenished jointly as much as possible. As we have discussed previously, there are several possible reasons for joint replenishments, for example, joint setup costs, quantity discounts, coordinated transports, etc.

The methods for determination of batch quantities and reorder points that we have dealt with in Chapters 4 - 6 are evidently not directly applicable in case of joint replenishments. If there is a joint setup cost, for example, we wish to have a total order size that is sufficiently large while the individual lot sizes may be of less importance. We would also like to choose the batch sizes such that we have reasonable possibilities to coordinate future orders. Furthermore, we can normally not use reorder points that have been determined individually. Assume that the individual reorder points correspond to a certain service level that we would like to maintain. If we make a joint replenishment as soon as one of the items reaches its reorder point, other items will be ordered too early. This means that the service level (and the holding costs) will be higher than what was intended.

7.3.1 A deterministic model

A common way of modeling joint replenishment problems is to assume that there are two types of ordering or setup costs: individual setup costs for each item, and a joint setup cost for the whole group of items. The joint setup cost does not depend on the number of items that are ordered. We shall first consider such a model with constant continuous demand. No backorders are allowed. The cycle times, or equivalently the batch quantities, are constant. The production time can be disregarded. We can also without any lack of generality disregard the lead-times, provided they are the same for all items. Let us introduce the following notation:

N = number of items,
h_i = holding cost per unit and time unit for item i,
A = setup cost for the group,
a_i = setup cost for item i,
d_i = demand per time unit for item i,
T_i = cycle time for item i.

The problem is to determine cycle times in order to minimize the sum of holding and setup costs. Given a cycle time T_i, the corresponding batch quantity Q_i is obtained as $Q_i = T_i d_i$. To simplify the notation we shall replace the holding costs h_i by $\eta_i = h_i d_i$, and set all demands equal to one. It is easy to check that this will not change the problem. (See Problem 7.12.) Furthermore, we assume for simplicity, that the items are ordered so that $a_1/\eta_1 \leq a_2/\eta_2 \leq \ldots \leq a_N/\eta_N$. We shall consider two approaches to solve the problem. The first approach is a simple iterative technique. The second one is based on Roundy's 98 percent approximation.

7.3.1.1 Approach 1. An iterative technique

If there were no joint setup cost the optimal cycle times could be determined by the classical economic lot size formula as:

$$T_i = \sqrt{\frac{2a_i}{\eta_i}}, \tag{7.44}$$

i.e., T_1 would be the smallest cycle time. A natural approach to solving the problem is therefore to assume that the cycle times of items 2, 3, ... , N are integer multiples n_i of the cycle time for item 1, or equivalently,

$$T_i = n_i T_1, \quad i = 2, 3, \ldots, N. \tag{7.45}$$

Our objective is then to minimize the total costs per time unit,

$$C = \frac{A + a_1 + \sum_{i=2}^{N} a_i / n_i}{T_1} + \frac{T_1(\eta_1 + \sum_{i=2}^{N} \eta_i n_i)}{2}, \tag{7.46}$$

with respect to T_1, n_2, n_3, ... , n_N.

Given n_2, n_3, ... , n_N, the optimal T_1 and the corresponding costs are obtained as:

$$T_1^*(n_2, n_3, \ldots, n_N) = \sqrt{\frac{2(A + a_1 + \sum_{i=2}^{N} a_i / n_i)}{\eta_1 + \sum_{i=2}^{N} \eta_i n_i}}, \tag{7.47}$$

$$C^*(n_2, n_3, \ldots, n_N) = \sqrt{2(A + a_1 + \sum_{i=2}^{N} a_i / n_i)(\eta_1 + \sum_{i=2}^{N} \eta_i n_i)}. \tag{7.48}$$

Note that T_1^* is not chosen according to (7.44).

If we disregard that n_2, n_3, ... , n_N have to be integers and optimize the costs (7.48), it is possible to show (Problem 7.13) that:

$$n_i = \sqrt{\frac{a_i}{\eta_i} \frac{\eta_1}{(A + a_1)}}. \tag{7.49}$$

Inserting (7.49) in (7.48) we get the following lower bound for the costs:

$$\underline{C} = \sqrt{2(A + a_1)\eta_1} + \sum_{i=2}^{N} \sqrt{2a_i\eta_i}. \tag{7.50}$$

The lower bound (7.50) can also be understood in the following way. Assume that there is no joint setup cost but that the individual setup cost for item 1 is $A + a_1$. If we optimize each item separately we obtain the total costs (7.50). But the considered problem is clearly a relaxation of the original problem since we can have setups of items 2, 3, ... , N without any joint setup cost. Consequently (7.50) is a lower bound for the total costs.

We are now ready to formulate a heuristic for the original problem where $n_2, n_3, \ldots, n_N$ are integers.

1. Determine start values of $n_2, n_3, \ldots, n_N$ by rounding (7.49) to the closest positive integers.

2. Determine the corresponding T_1 from (7.47).

3. Given T_1, minimize (7.46) with respect to $n_2, n_3, \ldots, n_N$. This means that we are choosing n_i as the smallest positive integer satisfying:

$$n_i(n_i + 1) \geq \frac{2a_i}{\eta_i T_1^2}. \tag{7.51}$$

   Return to Step 2 if any multiplier n_i has changed since the last iteration.

Since the procedure gives an improvement in each step it will obviously converge, but not necessarily to the optimal solution. The resulting costs can be compared to the lower bound (7.50).

Example 7.2 Consider $N = 4$ items with a joint setup cost $A = 300$. The individual setup costs are $a_1 = a_2 = a_3 = a_4 = 50$, and the holding costs are $h_1 = h_2 = h_3 = h_4 = 10$. The demands per time unit are $d_1 = 5000$, $d_2 = 1000$, $d_3 = 700$, and $d_4 = 100$. Note that in this case $\eta_i = h_i d_i = 10d_i$. As requested, a_i/η_i is nondecreasing with i. When applying the heuristic we obtain:

1. $n_2 = \sqrt{\frac{50}{1000 \cdot 10} \frac{5000 \cdot 10}{350}} \approx 1, n_3 \approx 1, n_4 \approx 3$.

2. $T_1 = \sqrt{\frac{2 \cdot (350 + 50 + 50 + 50/3)}{10 \cdot 5000 + 10 \cdot 1000 + 10 \cdot 700 + 10 \cdot 100 \cdot 3}} = 0.1155$.

3. We again obtain the multipliers $n_2 = 1$, $n_3 = 1$, $n_4 = 3$, i.e., the algorithm has already converged.

We get the resulting costs from (7.48) as $C = 8082.9$, which can be compared to the lower bound according to (7.50) $\underline{C} = 8069.0$.

Related procedures are described in e.g., Goyal and Satir (1989), and Silver et al. (1998). A technique for finding a solution with an arbitrarily small deviation from the optimal value is given by Wildeman et al. (1997).

7.3.1.2 Approach 2. Roundy's 98 percent approximation

We shall now look for a solution where the joint setups have cycle time $T_0 \geq 0$, and all other cycle times are nonnegative powers-of-two times T_0, i.e.,

$$T_i = 2^{k_i} T_0, \qquad i = 1, 2, ..., N, \tag{7.52}$$

where k_i is a nonnegative integer. Using the notation $a_0 = A$, and $\eta_0 = 0$, we can express our objective as:

$$\min_{T_0, T_1, \ldots, T_N} \sum_{i=0}^{N} \left(\eta_i \frac{T_i}{2} + a_i \frac{1}{T_i} \right), \tag{7.53}$$

subject to the constraints (7.52). This means that we let the joint setups be represented by a fictive item 0.

Let us now relax the considered problem by replacing (7.52) by

$$T_i \geq T_0, \qquad i = 1, 2, \ldots, N. \tag{7.54}$$

It is a relaxation because (7.52) implies (7.54) while the opposite is not true. The resulting solution will therefore give a lower bound for the costs.

Consider the relaxed problem, i.e., (7.53), with respect to the constraints (7.54). Since the objective function is convex and the constraints are linear, we can get the solution from the following Lagrangean relaxation:

$$\max_{\lambda_1,\lambda_2,\ldots,\lambda_N} \min_{T_0,T_1,\ldots,T_N} \sum_{i=0}^{N}\left(\eta_i \frac{T_i}{2} + a_i \frac{1}{T_i}\right) + \sum_{i=1}^{N} \lambda_i (T_0 - T_i), \qquad (7.55)$$

where the multipliers λ_i are also required to be nonnegative. Let us define:

$$\eta_0' = \eta_0 + 2\sum_{i=1}^{N} \lambda_i ,$$

$$\eta_1' = \eta_1 - 2\lambda_1,$$

$$\vdots \qquad (7.56)$$

$$\eta_N' = \eta_N - 2\lambda_N.$$

It is easy to see that the optimal solution must have all $\eta_i' > 0$. Otherwise $T_i \to \infty$, which is obviously not optimal.

Given the multipliers, the optimal solution of the relaxed problem can be obtained by solving:

$$\min_{T_0,T_1,\ldots,T_N} \sum_{i=0}^{N}\left(\eta_i' \frac{T_i}{2} + a_i \frac{1}{T_i}\right), \qquad (7.57)$$

without any constraints on the cycle times, i.e., we have $N + 1$ independent classical lot sizing problems.

Let us now go back to the original joint replenishment problem but with cost parameters a_i and η_i'. Consider the effect of λ_i on the holding costs. The holding cost of item i is reduced by $2\lambda_i$ and the holding cost of item 0 is increased by $2\lambda_i$. This will mean a cost reduction for any solution of the joint replenishment problem, even if we allow the periods between orders to vary over time. To see this, consider the period between two orders for item $i > 0$. If there are additional joint setups between the two orders the total holding costs during the considered period will decrease, otherwise they will be the same.

But given the cost parameters a_i and η_i', (7.57) will obviously provide the minimum cost since no constraints are considered. The optimal solution of the relaxed problem will consequently give a lower bound for the costs of any solution. This bound is tighter than (7.50) because of the constraints (7.54).

Assume that we have solved the relaxed problem. We can then adjust this solution by rounding the cycle times so that they can be expressed as:

$$T_i = 2^{m_i} q \tag{7.58}$$

for some number $q > 0$. We know from Proposition 7.1 in Section 7.1 that if q is given, the maximum cost increase is at most 6 percent, and if we can also adjust q to get a better approximation, the cost increase is at most 2 percent according to Proposition 7.2. Due to (7.54), we know that $m_i \geq m_0$ and the cycle times obtained must consequently satisfy (7.52). We have now obtained Roundy's solution of the problem. This solution has an important quality. The cost increase compared to the optimal solution is at most 2 percent, since it is at most 2 percent compared to the lower bound (7.57).

It is possible to use the considered Lagrangean relaxation for numerical determination of Roundy's solution, but in general, it is much simpler to use the following technique. Since the items are ordered so that a_i/η_i are nondecreasing with i for $i > 0$, it is obvious from (7.53) and (7.54) that the optimal cycle times in the relaxed problem are nondecreasing with i for $i > 0$. It is also clear that we must have $T_1 = T_0$ in the optimal solution of the relaxed problem because $a_0 = A$, and $\eta_0 = 0$. This means that

$$T_i \geq T_{i-1}, \qquad i = 1, 2, \ldots, N. \tag{7.59}$$

Consider the relaxed problem with (7.54) replaced by (7.59). This will not change the optimal solution. Without the constraints (7.59) it would be optimal to use $T_i^* = (2a_i/\eta_i)^{1/2}$ for all i. Consequently, if a_i/η_i is increasing with i, we have found the optimal solution since the resulting batch quantities will satisfy (7.59). Since $\eta_0 = 0$ this is never the case initially. Assume that for some i, $a_i/\eta_i < a_{i-1}/\eta_{i-1}$, or equivalently that $T_i^* < T_{i-1}^*$. Assume furthermore that in the optimal solution $T_i > T_{i-1}$. Because of the convexity this implies that $T_i \leq T_i^*$ since we would otherwise reduce T_i, and similarly that $T_{i-1} \geq T_{i-1}^*$ since we would otherwise increase T_{i-1}. But this means that $T_i < T_{i-1}$ which is a contradiction. Consequently, $T_i = T_{i-1}$ in the optimal solution of the relaxed problem. But this implies that we can aggregate items i - 1 and

i into a single item with cost parameters $a_{i-1} + a_i$, and $\eta_{i-1} + \eta_i$. Next we consider the resulting reduced problem with one item less. If $a_i/\eta_i < a_{i-1}/\eta_{i-1}$ for some i we can aggregate the two items, otherwise we obtain the optimal solution from the classical economic lot size model, etc.

Since $\eta_0 = 0$ we will always aggregate items 0 and 1. After aggregation we have an item with cost parameters $A + a_1$ and η_1. Next we check whether $a_2/\eta_2 < (A + a_1)/\ \eta_1$. If this is the case the aggregate item should include also item 2, etc. When no more aggregations are possible, we can optimize the resulting aggregate items individually.

Example 7.3 Consider the same data as in Example 7.2, i.e., $N = 4$, $A = 300$, $a_1 = a_2 = a_3 = a_4 = 50$, $\eta_1 = 50000$, $\eta_2 = 10000$, $\eta_3 = 7000$, and $\eta_4 = 1000$.

To solve the relaxed problem we first aggregate items 0 and 1. The combined item has cost parameters $A + a_1 = 350$, and $\eta_0 + \eta_1 = 50000$. Consider then item 2. Since $a_2/\eta_2 = 50/10000 < 350/50000$, item 2 should also be added to the combined item. The resulting cost parameters are obtained as $A + a_1 + a_2 = 400$, and $\eta_0 + \eta_1 + \eta_2 = 60000$. Consider item 3. Since $50/7000 > 400/60000$, item 3 should not be added. Compare finally item 4 and item 3. We get $50/1000 > 50/7000$, i.e., items 3 and 4 should not be combined. This gives the cycle times $T_0 = T_1 = T_2 = (800/60000)^{1/2} = 0.1155$, $T_3 = (100/7000)^{1/2} = 0.1195$, and $T_4 = (100/1000)^{1/2} = 0.3162$. The resulting lower bound for the costs is $\underline{C} = C_{0+1+2} + C_3 + C_4 = 6928.2 + 836.7 + 316.2 = 8081.1$. Note that this bound is better than the bound obtained in Example 7.2.

Consider then cycle times that can be expressed as powers of two, i.e., let $q = 1$ in (7.58). We obtain $T_0 = T_1 = T_2 = 2^{-3} = 0.125$, $T_3 = 2^{-3} = 0.125$, and $T_4 = 2^{-2} = 0.250$. The resulting costs are $C = C_{0+1+2} + C_3 + C_4 = 6950 + 837.5 + 325 = 8112.5$, i.e., 0.39 percent above the lower bound.

By minimization of (7.10) it is possible to show that it is optimal to have $q = 1.88$ in (7.58). The corresponding cycle times are $T_0 = T_1 = T_2 = 2^{-4}q = 0.1175$, $T_3 = 2^{-4}q = 0.1175$, and $T_4 = 2^{-3}q = 0.235$. The resulting costs are $C = C_{0+1+2} + C_3 + C_4 = 6929.3 + 836.8 + 330.3 = 8096.3$, i.e., 0.19 percent above the lower bound. The solution in Example 7.2, which is not a powers-of-two policy, is still slightly better.

A similar approach is used for multi-stage lot sizing in Section 9.2.2. See also Jackson et al. (1985), Roundy (1985, 1986), and Muckstadt and Roundy (1993).

The two approaches considered assume constant demand, but can also be used in case of stochastic demand. We then replace the stochastic demands by their means when determining cycle times. Given the cycle times we can, using the techniques in Section 5.12, determine appropriate periodic review S policies for each item. The order-up-to inventory positions should include suitable amounts of safety stock. Next, in Section 7.3.2, we will consider a different model that is more directly focused on stochastic demand.

7.3.2 A stochastic model

We shall now instead consider a stochastic model. The demands for the items are independent and stationary stochastic processes. Each customer demand is for an integral number of units. We can, for example, consider Poisson or compound Poisson demand processes. We assume complete backordering. Let us introduce the following notation to describe the problem:

N = number of items,
h_i = holding cost per unit and time unit for item i,
$b_{1,i}$ = shortage cost per unit and time unit for item i,
A = setup cost for the group,
a_i = setup cost for item i,
L = constant lead-time.

Viswanathan (1997) suggests the following technique that, in his numerical tests, outperforms other suggested methods.

In the first step we disregard the joint setup cost and consider the items individually for a suitable grid of review periods T. For each review period we determine the optimal individual (s, S) policies for all items and the corresponding average costs. This can be done very efficiently as explained in Section 6.1.1.2 Let

$C_i(T)$ = average costs per time unit for item i when using the optimal individual (s, S) policy with a review interval of T time units.

In the second step we determine the review period T by minimizing,

$$C(T) = A/T + \sum_{i=1}^{N} C_i(T). \tag{7.60}$$

Note that the actual costs are lower than the costs according to (7.60), since the major setup cost A is not incurred at reviews where none of the items are ordered.

Atkins and Iyogun (1988) also use periodic review policies but in a different way.

Other policies that have been used frequently in the inventory literature are so-called *can-order policies*. When using such a policy there are two reorder points for each item: a *can-order* level and a lower *must-order* level. An item must be ordered when its inventory position reaches the must-order level. When an item in the group is ordered, other items with inventory positions at or below their respective can-order levels are also ordered. This type of policy was first suggested by Balintfy (1964). Techniques for designing can-order policies have been suggested by Silver (1981) and Federgruen et al. (1984).

Renberg and Planche (1967) suggested a so-called (S_i, Q) policy. According to this policy all items are replenished up to certain levels S_i when the total demand for the whole group since the preceding replenishment has reached Q.

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Problems

7.1 Show that (7.4) implies (7.5).

7.2 a) Show that $\int_{-1/2}^{1/2} e(u)du = \frac{1}{\sqrt{2}\ln 2}$.

b) Show that the worst case will occur in (7.9) if x is uniform on (- 1/2, 1/2). What happens if we change q? Why?

7.3* Three products are produced in a single machine. The demands are constant and continuous. No backorders are allowed. The following data are given

Product	Demand, units per day	Holding cost per unit and day	Setup cost	Setup time, days	Production rate, units per day
1	48	0.060	800	0.50	200
2	20	0.040	500	0.25	100
3	32	0.048	1000	1.00	100

a) Determine the independent solution and the corresponding lower bound for the optimal costs.
b) Show that the independent solution is infeasible.
c) Derive the common cycle solution and an upper bound for the total costs.
d) Apply Doll and Whybark's technique. Is the solution feasible?

7.4 Demonstrate that the iterative procedure in Section 7.2.1.5 will converge.

7.5* Three products are produced in the same machine. Various data are given in the table.

Product	I	II	III
Demand per week	100	50	20
Production per week	1000	500	250
Setup time in weeks	0.8	0.4	0.1
Ordering cost	10000	10000	5000
Holding cost per unit and week	10	20	10

a) Use a cyclic schedule with a common cycle. Determine batch quantities.
b) By using the technique by Doll and Whybark the following solution has been found:

$Q_1 = 460$
$Q_2 = 230$
$Q_3 = 184$

* Answer and/or hint in Appendix 1.

Does this solution reduce the costs, and in that case how much. Is the solution feasible?

7.6 A company is producing three products in the same machine. There are 250 working days in one year, and 8 working hours per day. The following data are given:

Product	Demand, units per year	Holding cost per unit and year	Setup cost	Setup time, minutes	Operation time, minutes per unit
1	8000	20	120	20	5
2	12000	15	100	10	3
3	5000	30	200	30	8

Determine the best cyclic plan with a common cycle time.

7.7 Prove Proposition 7.3.

7.8 Show that the set of *demand-feasible* production plans is convex.

7.9 Show that the number of plans satisfying *Property 1* is at most 2^{T-1}.

7.10 Consider Example 7.1.

a) Represent the following plan for item 1: (30, 40, 55) as a convex combination of plans satisfying *Property 1*. Is the solution unique?
b) Show that the plan obtained by the linear program is infeasible.
c) Demonstrate that the optimal solution of the problem is to use the plan (50, 0, 75) for item 1 and (20, 75, 0) for item 2.

7.11 Consider (7.42). Show that the average time in the system T is minimized by choosing Q according to (7.43).

7.12 Consider the model in Section 7.3.1. Show that we, without changing the problem, can replace the holding costs h_i by $\eta_i = h_i d_i$, and set all demands equal to one.

7.13 Derive the optimal continuous n_i, (7.49), from (7.48).

7.14 Derive the condition (7.51).

7.15 Start with the solution obtained in Example 7.3 and check whether the technique in Section 7.3.1.1 can improve the solution. This means that we no longer require a powers-of-two policy.